Cliques, Bicliques, Clubs and Colours

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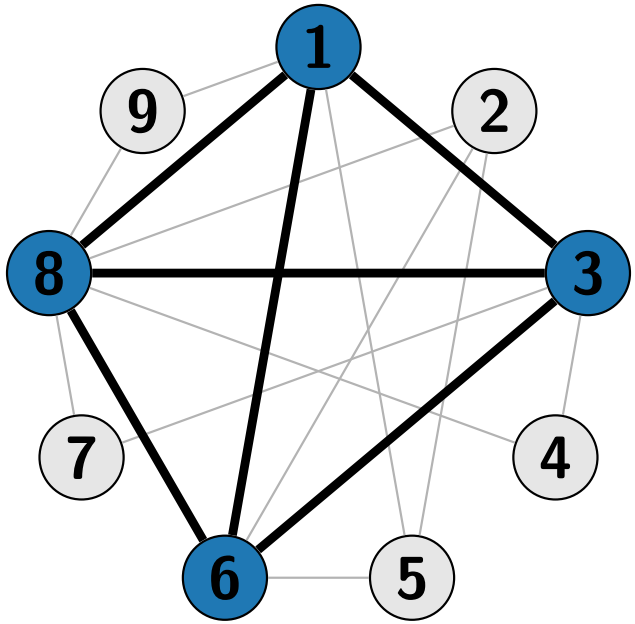
November 19, 2013

Cliques and Colours

The Maximum Clique Problem

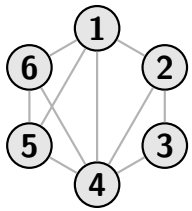


- Determining the size of a maximum clique, denoted ω, is NP-hard.

Branch and Bound for Maximum Clique

- Pick a vertex v. Now we *branch*: either v is in a maximum clique...
  - Add v to a growing candidate clique.
  - Discard undecided vertices not adjacent to v.
  - Recurse.
- ...or it is not. Exclude v, and try again until we're out of vertices.
- Keep track of the largest clique found so far (the *incumbent*), and avoid recursing if we cannot beat the incumbent. So we need a *bound* on the size of a maximum clique.

Branch and Bound for Maximum Clique



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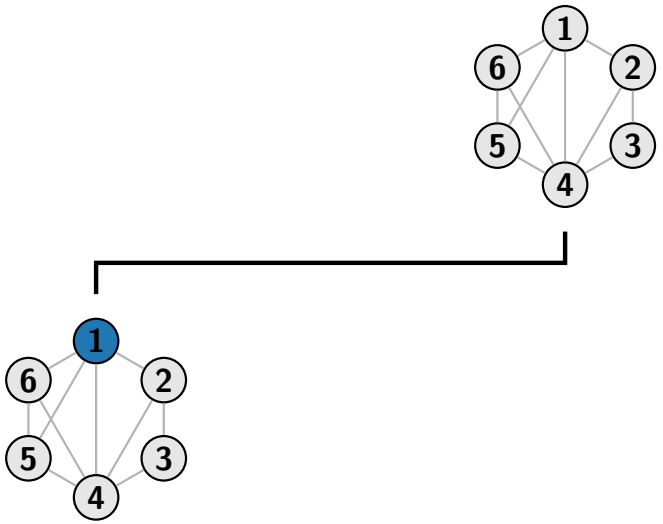

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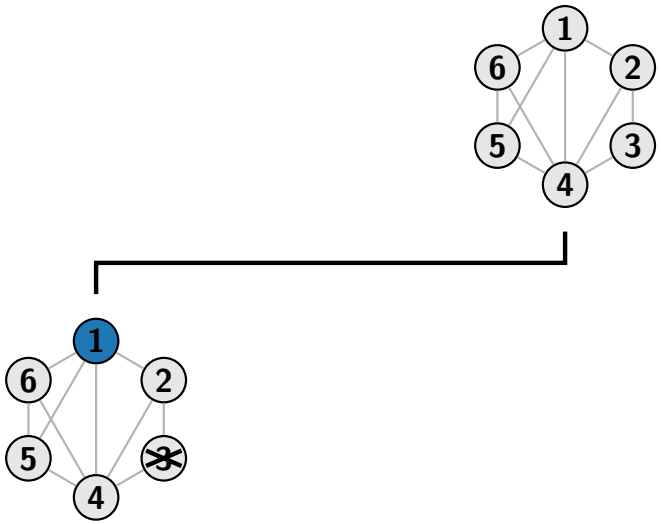

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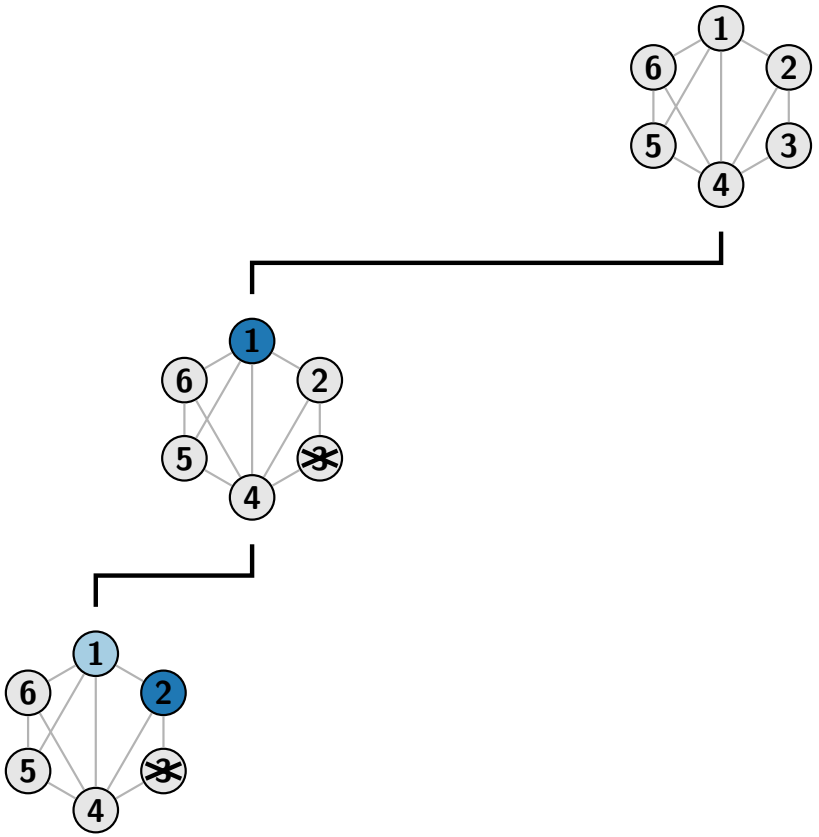

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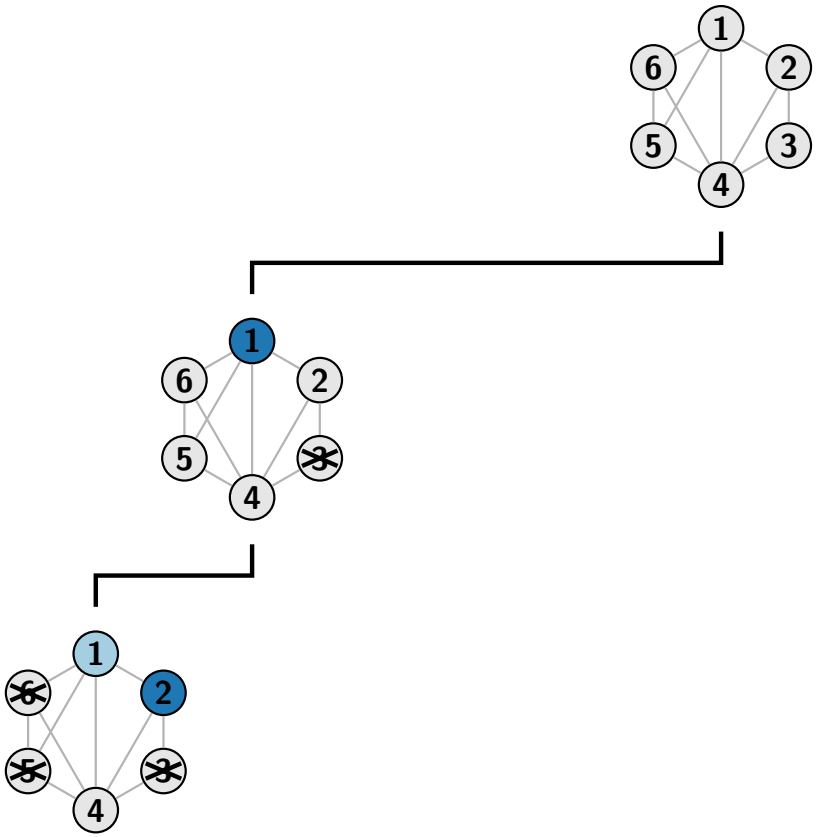

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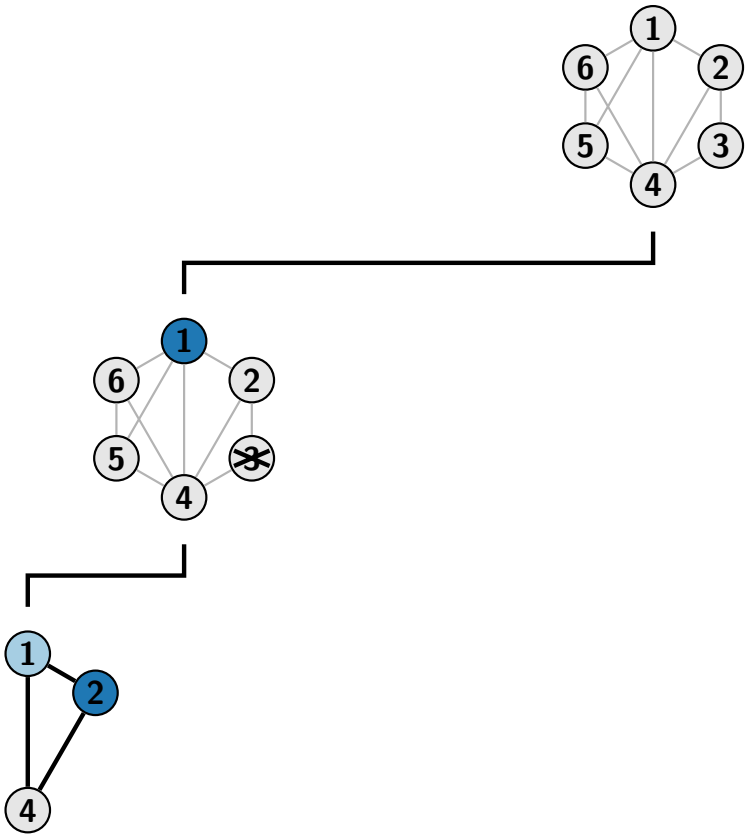

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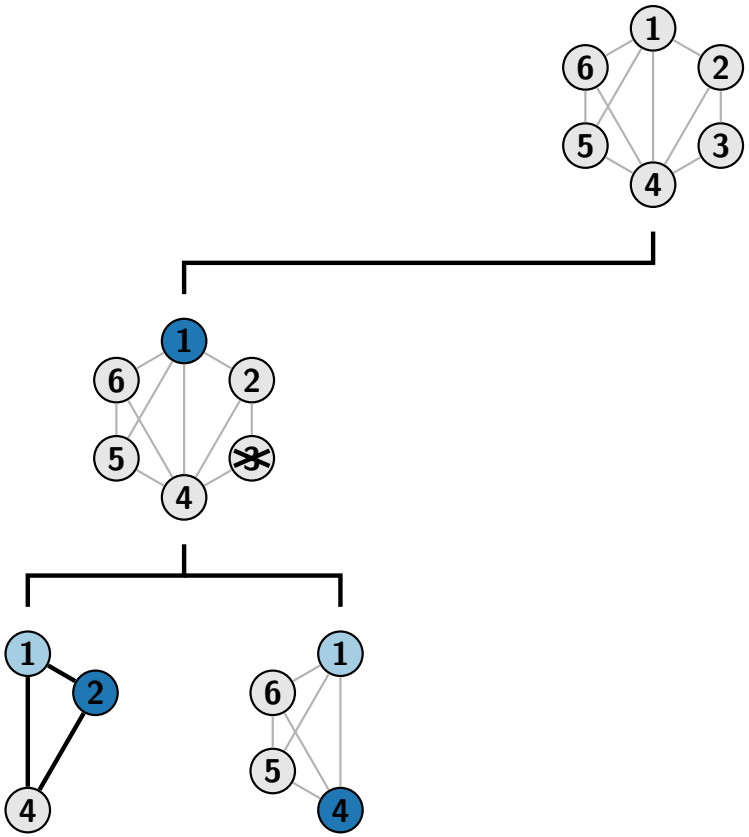

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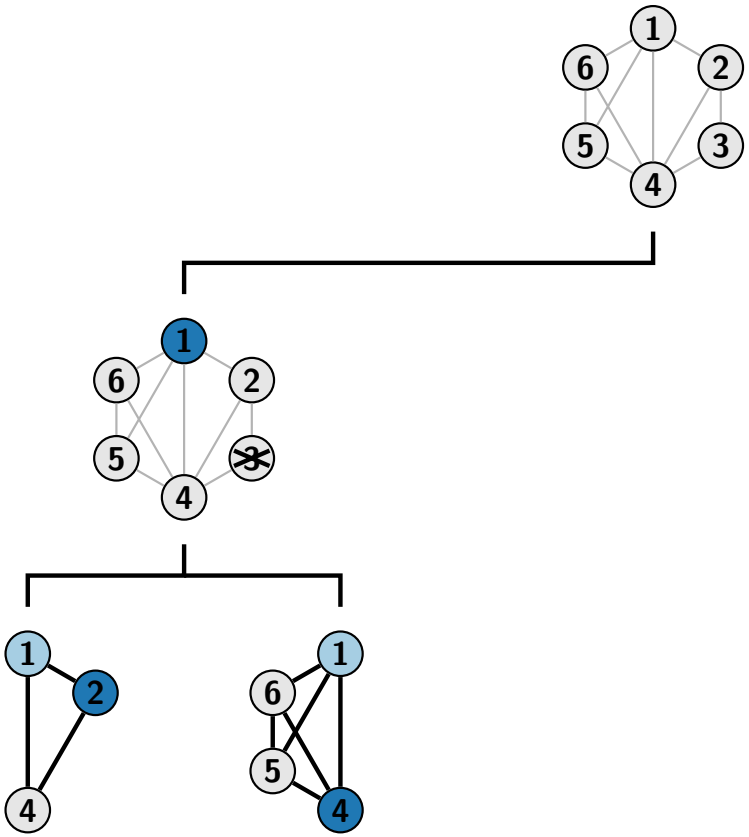

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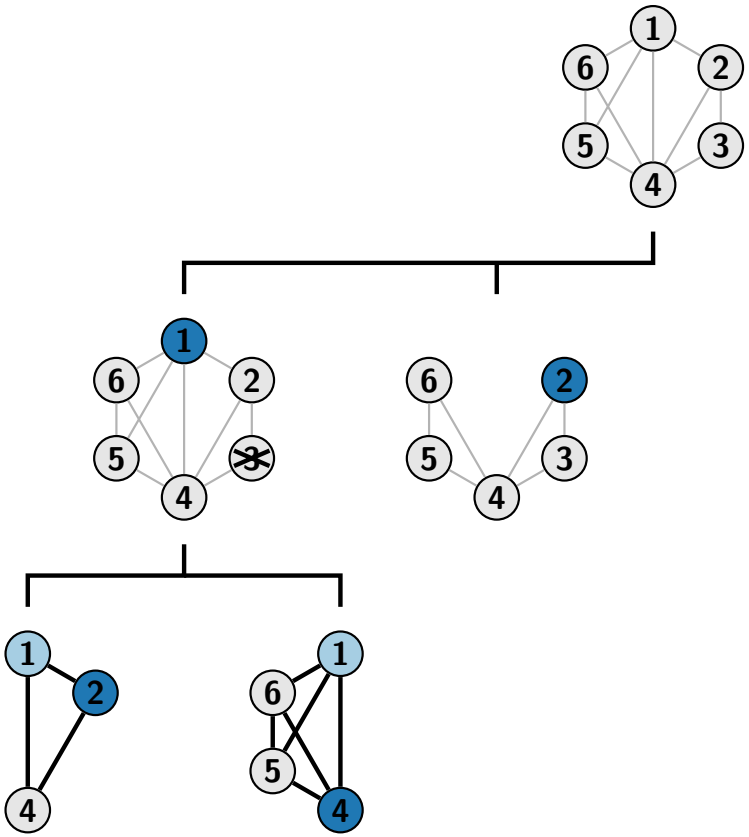

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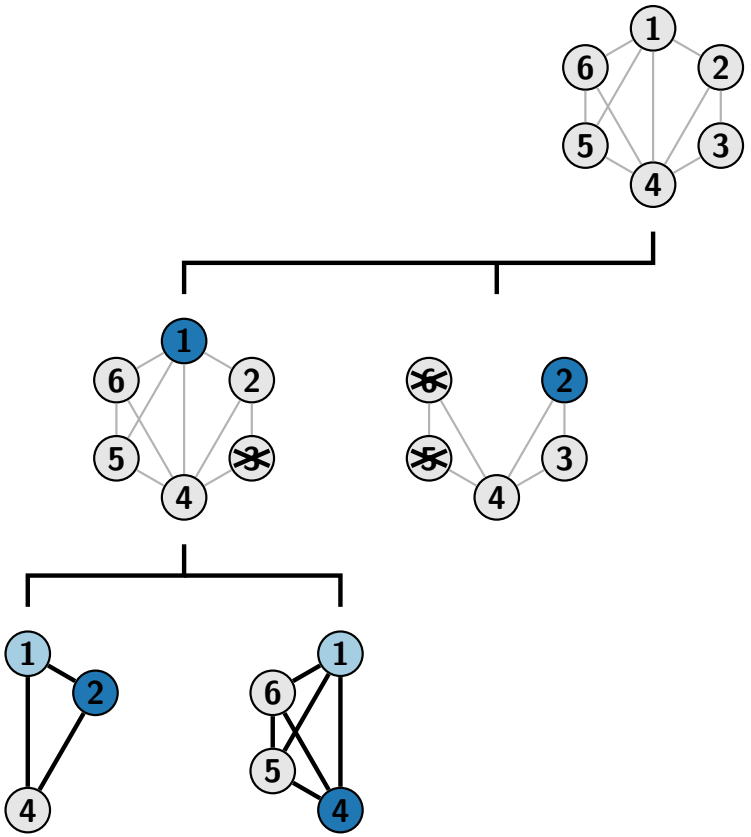

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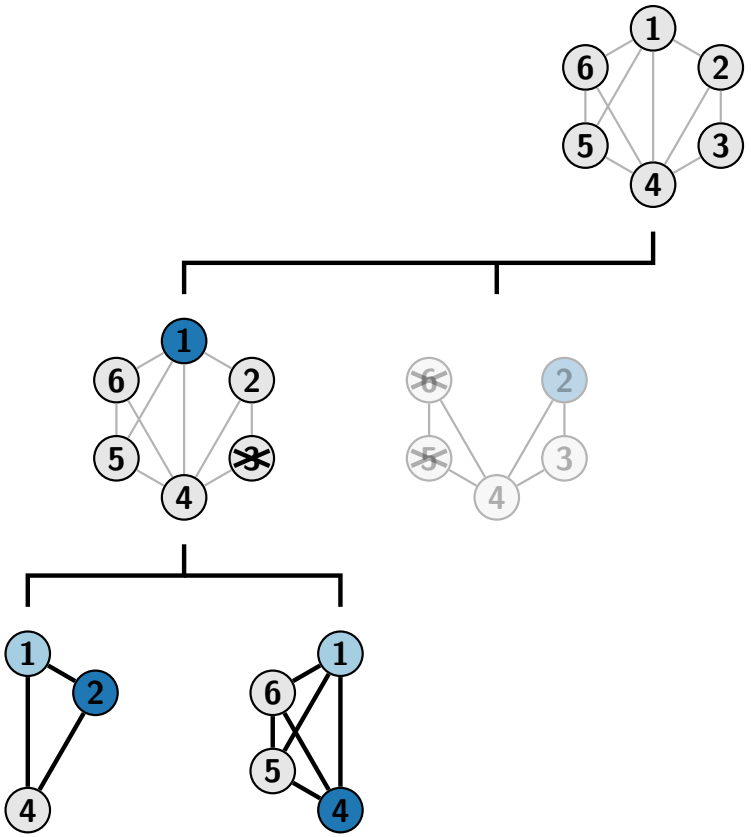

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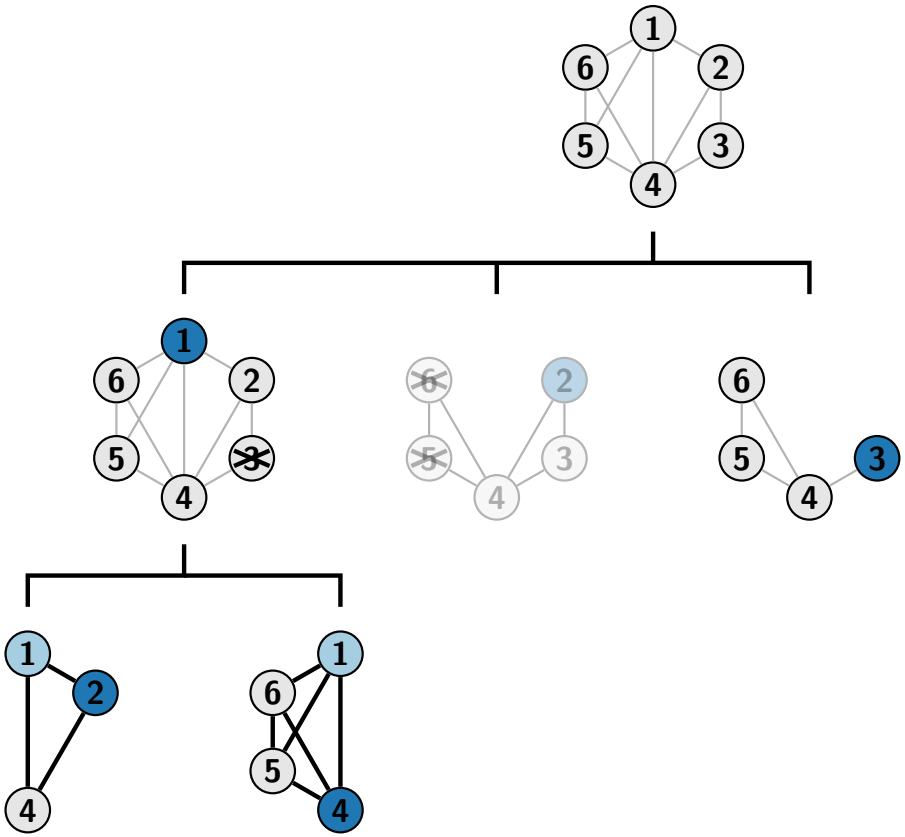

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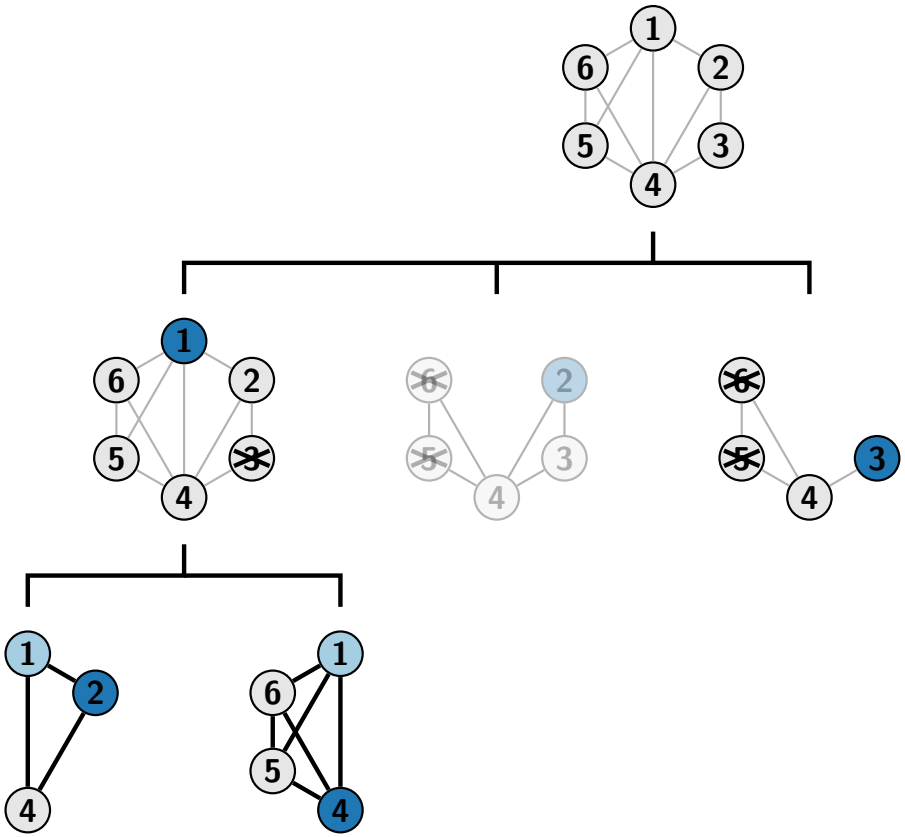

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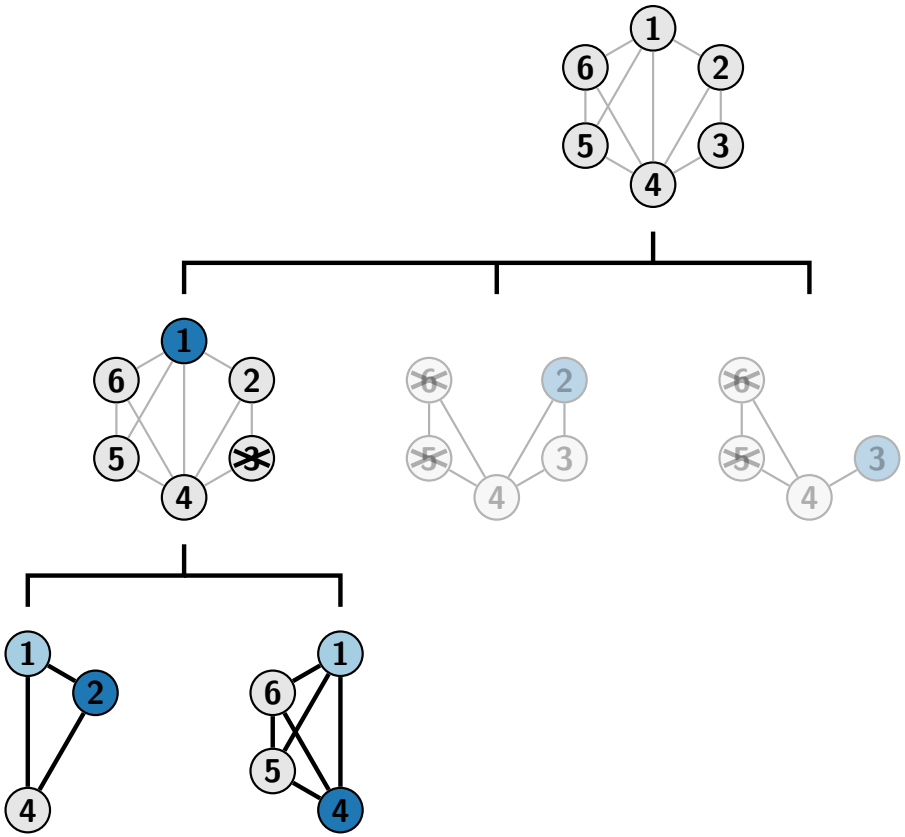

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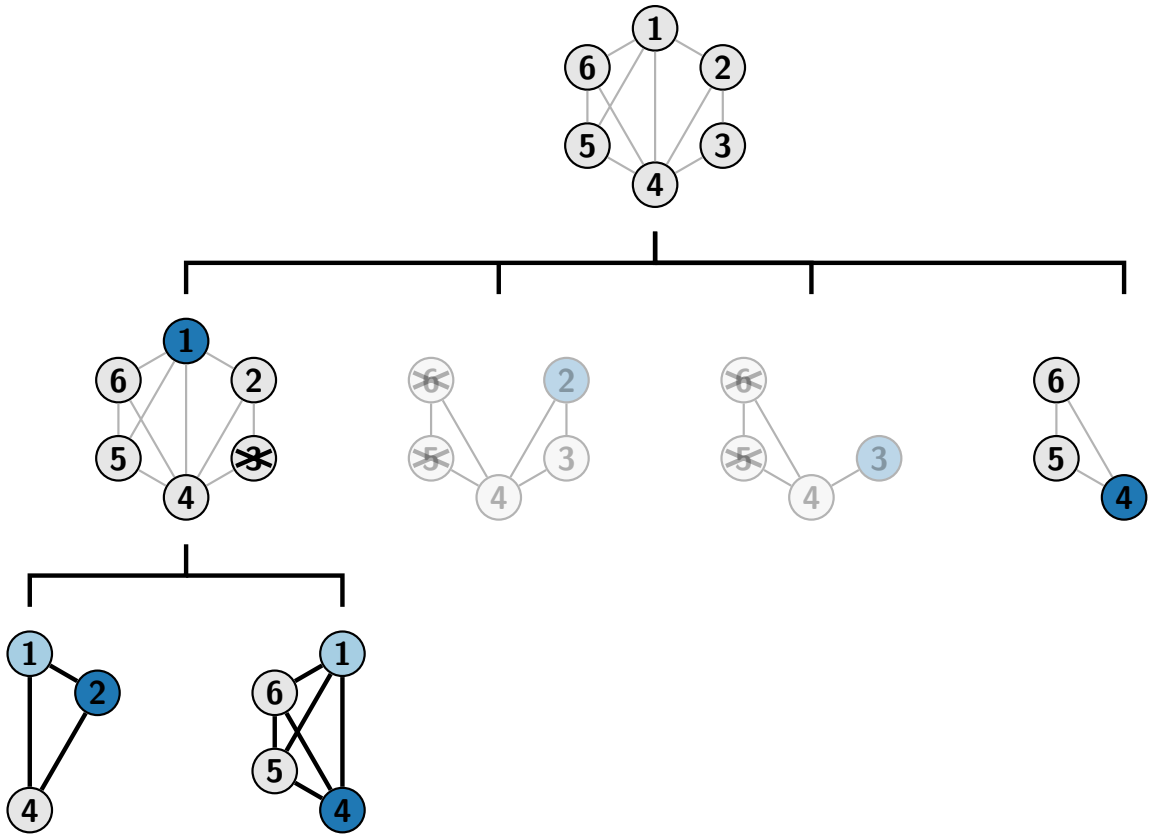

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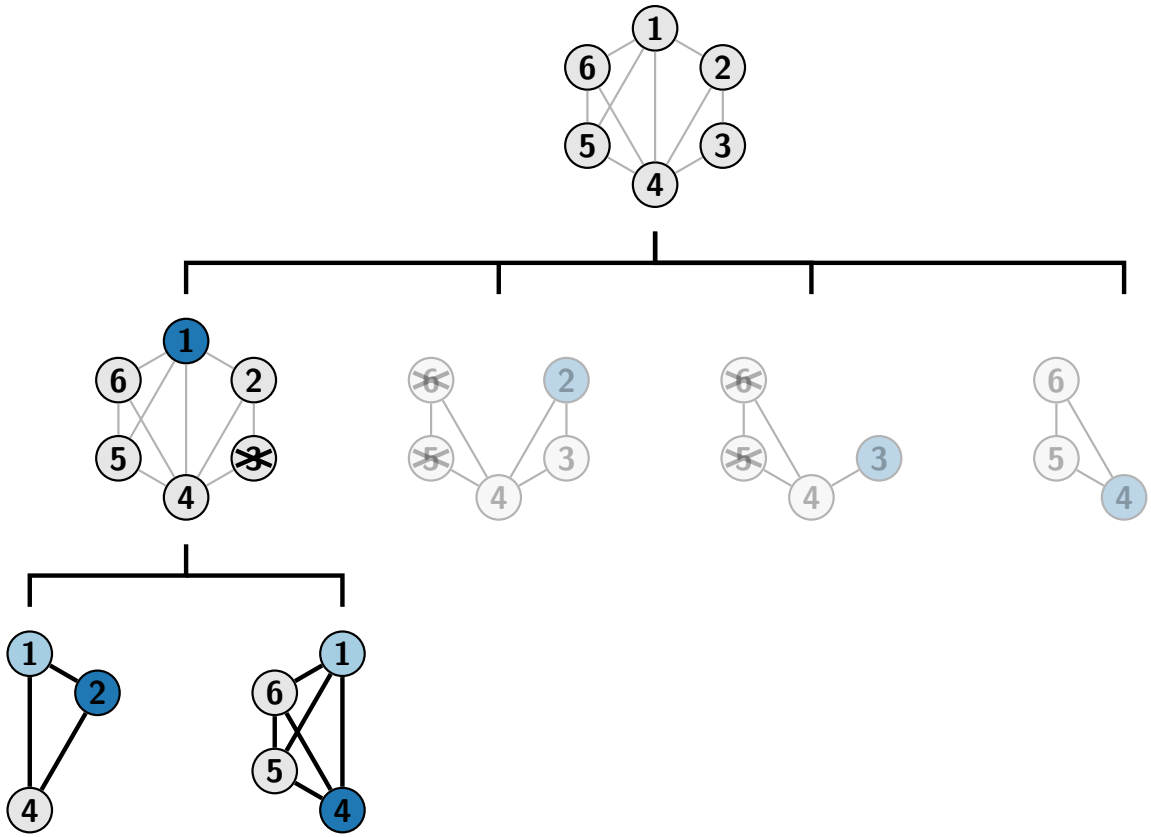

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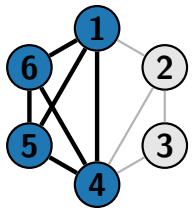
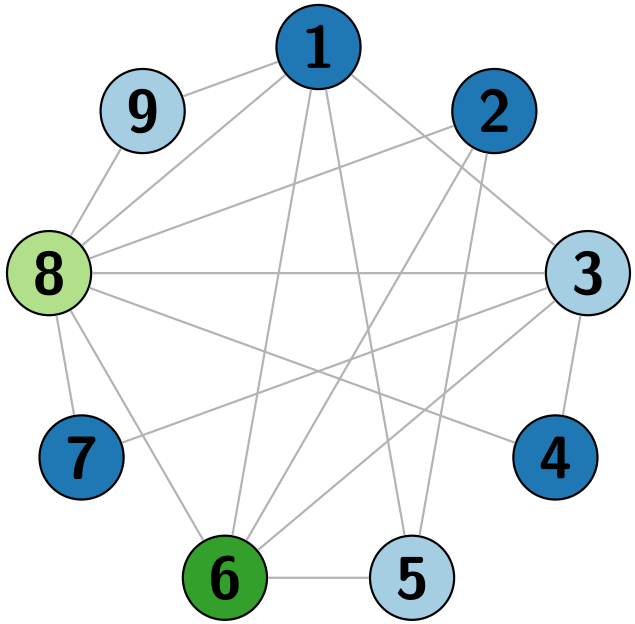


Bound?

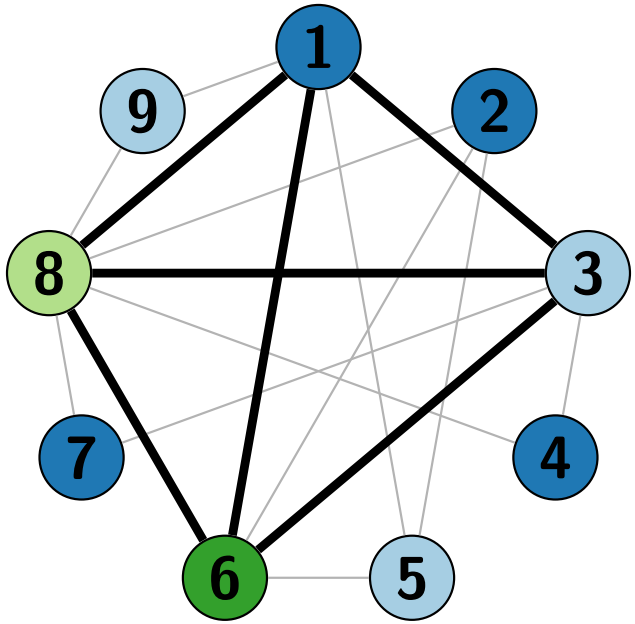
- $\omega \leq |V|$. But this is not very good.

Graph Colouring



- Determining the smallest number of colours needed, denoted χ, is NP-hard.

Graph Colouring

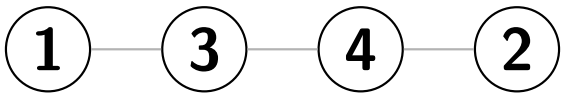


- Determining the smallest number of colours needed, denoted χ, is NP-hard.
- $\omega \leq \chi$. In this case we also know that any clique of size three must contain a green vertex, and any clique of size four must contain both green vertices.

Greedy Colouring

- We can construct a non-optimal colouring in polynomial time by colouring greedily:
  - Give each vertex in turn the first available colour
  - . . . or while there are vertices uncoloured, start a new colour, and assign each remaining vertex in turn that colour if possible.

Greedy Colouring is Bad and Misleading



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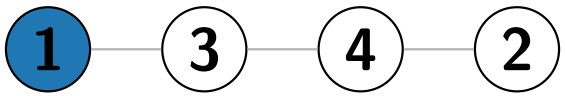

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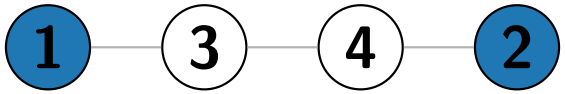

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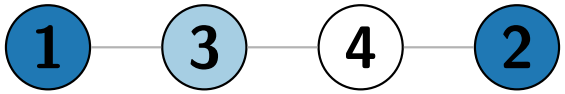

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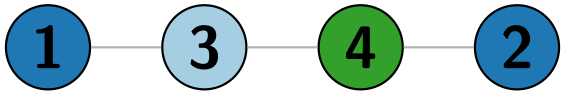

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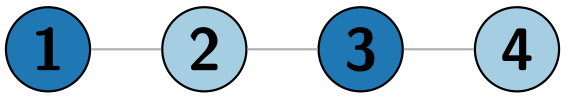

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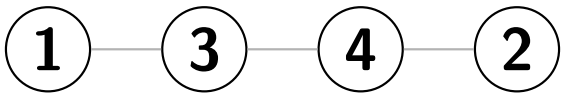

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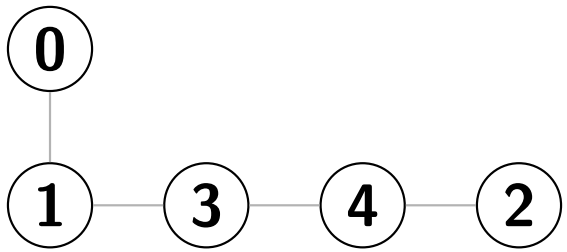

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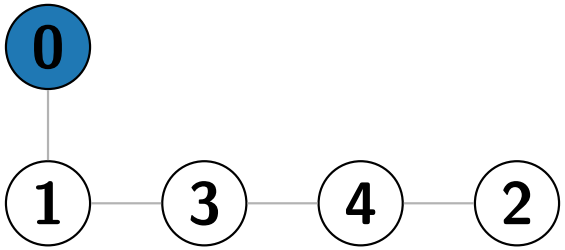

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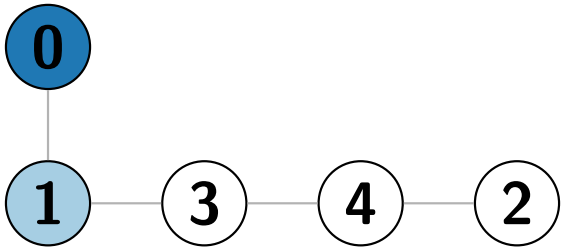

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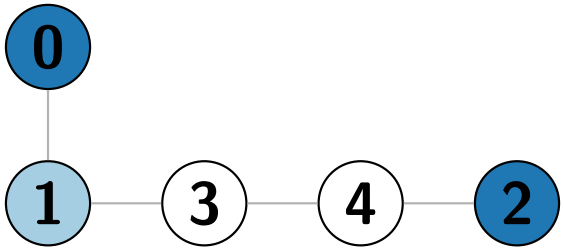

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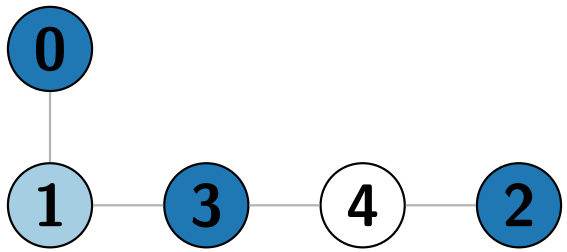

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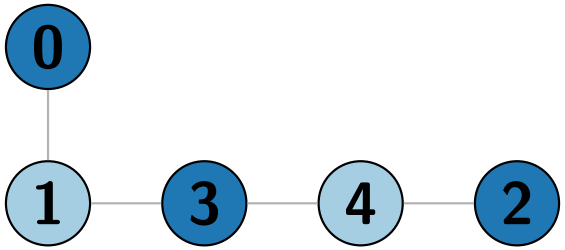

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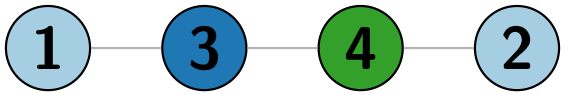
Fixing a Greedy Colouring?



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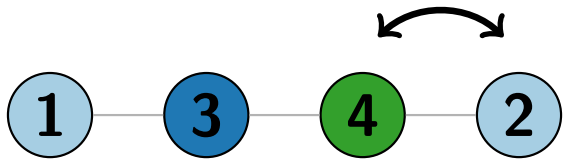

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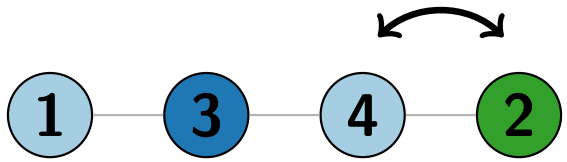

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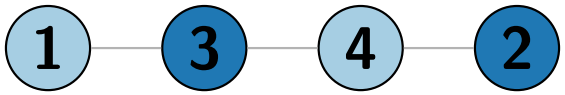

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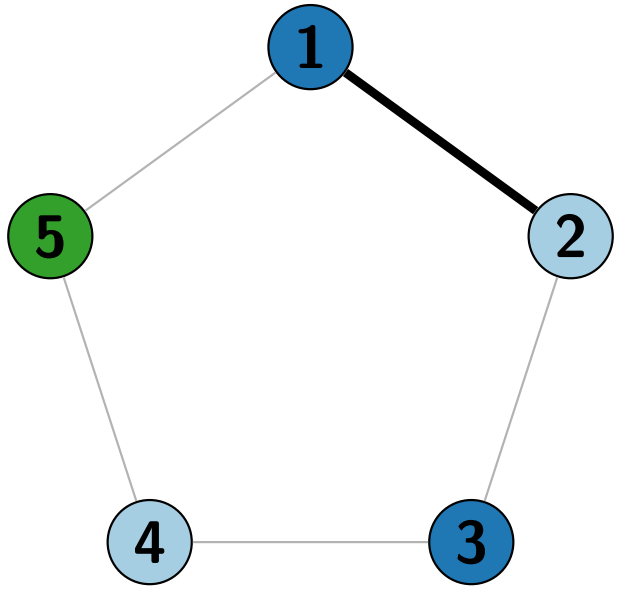
$\chi - \omega$ Can Be Large



- There are graphs where $\omega < \chi$.

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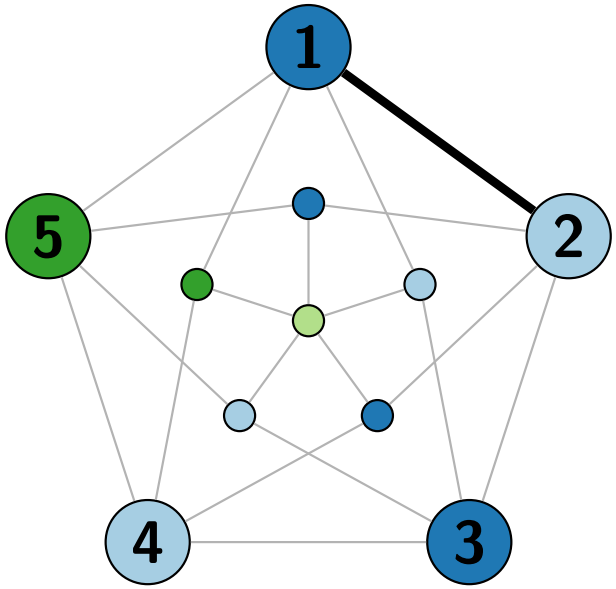
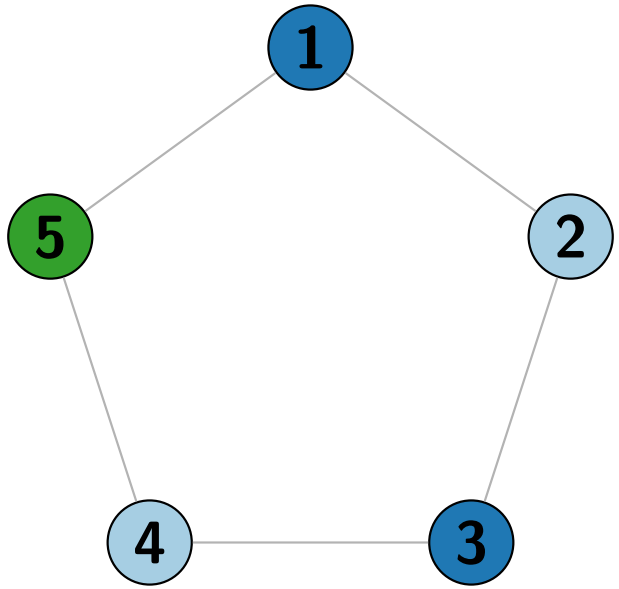


- There are graphs where $\omega < \chi$.
- There are graphs where $\chi - \omega$ is large. Testing whether $\chi - \omega = 1$ is NP-complete.

Measuring the Gap?



- If we pick the green vertex, we can't also pick both a light blue and a dark blue vertex.

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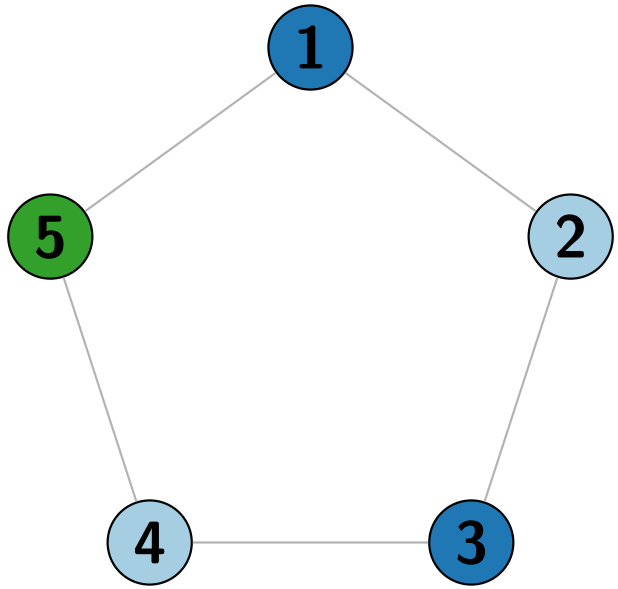

- If we pick the green vertex, we can't also pick both a light blue and a dark blue vertex.
- MaxSAT unit propagation can prove this, giving a lower bound on $\chi - \omega$.

Measuring the Gap?

Hard:

$(\overline{x_1} \vee \overline{x_3})$

$(\overline{x_1} \vee \overline{x_4})$

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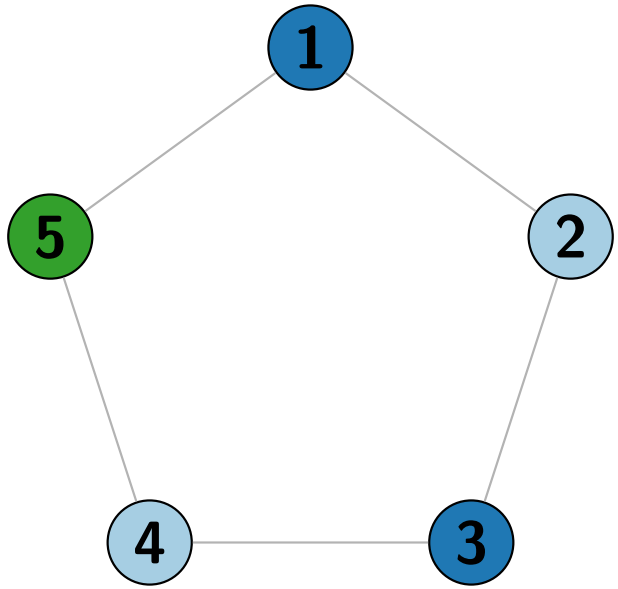

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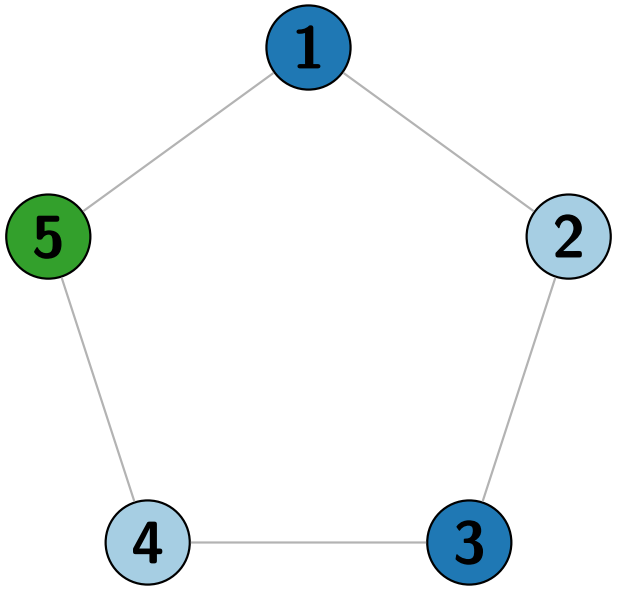

Soft:

$(x_1 \vee x_3)$

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(x_5)

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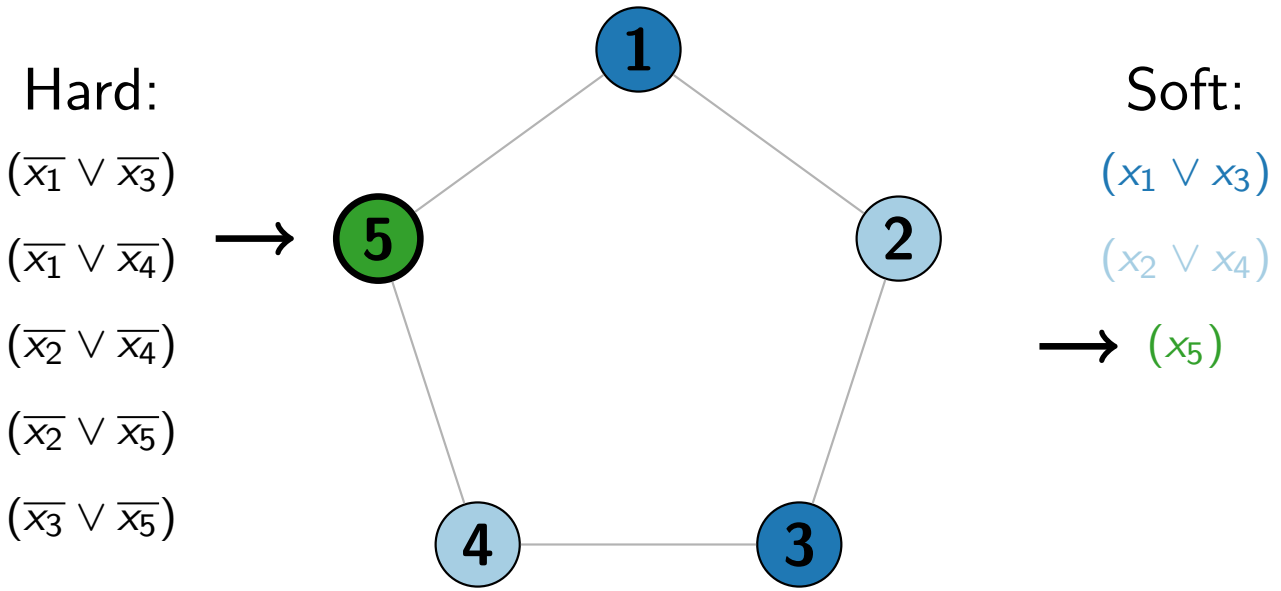

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$\longrightarrow$ $(\overline{x_2} \vee \overline{x_5})$

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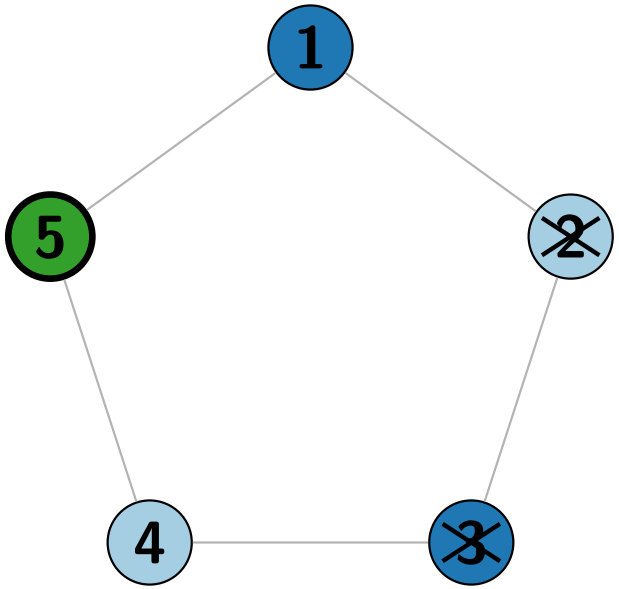

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- If we pick the green vertex, we can't also pick both a light blue and a dark blue vertex.
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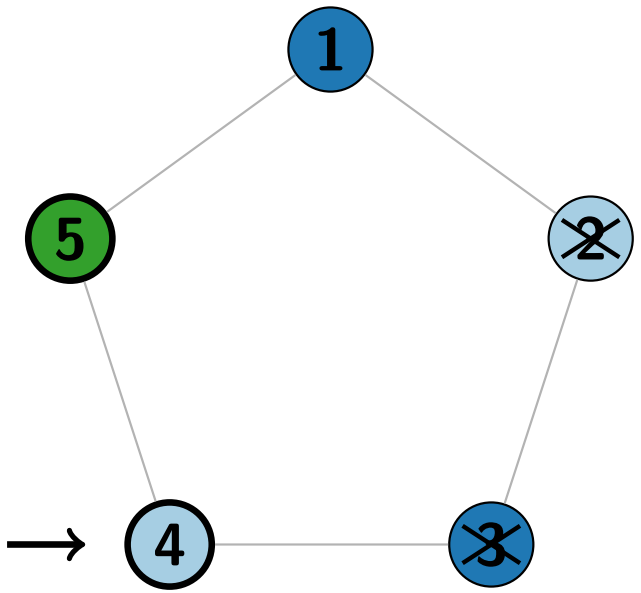

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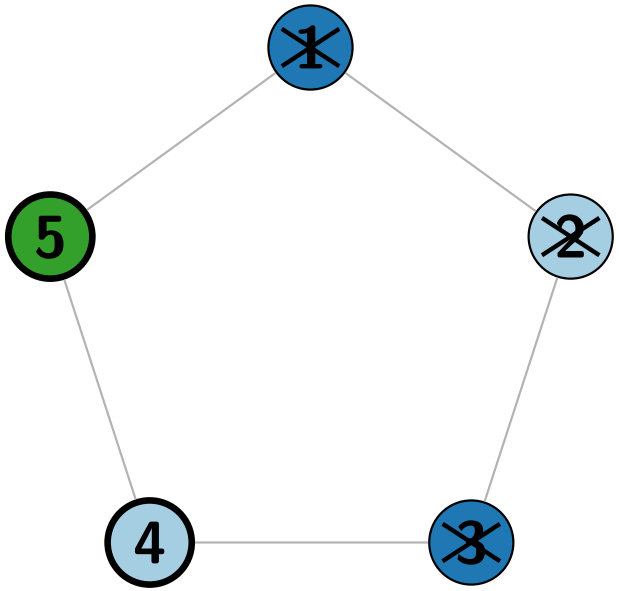

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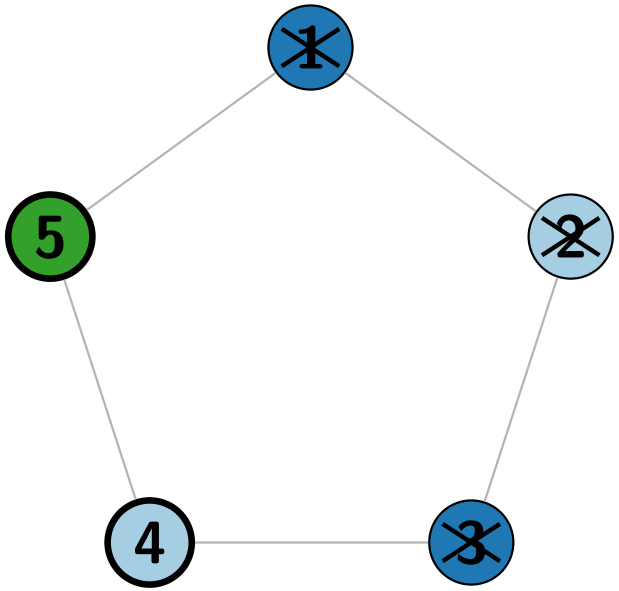

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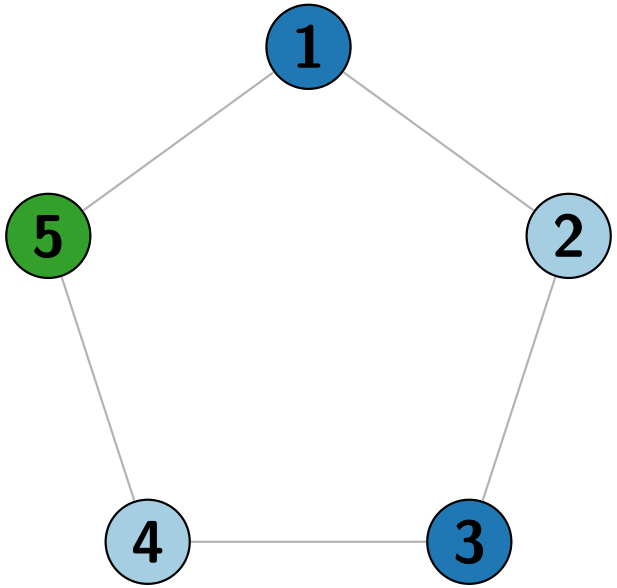
Fractional Colouring?



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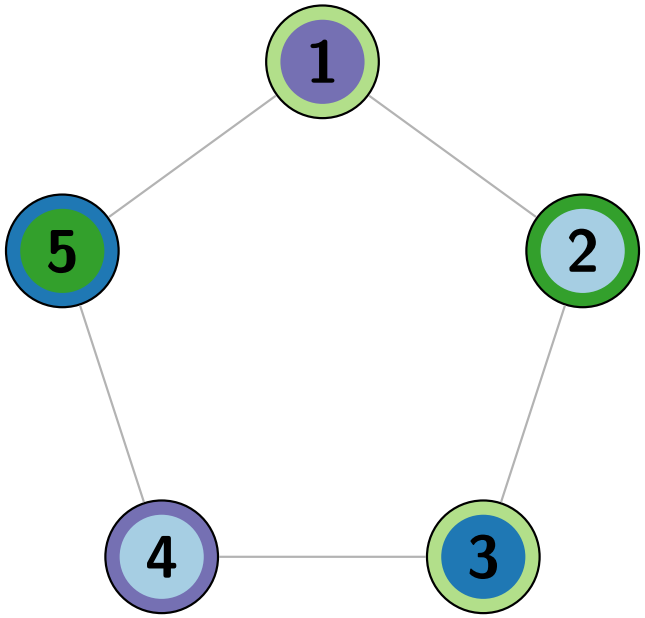

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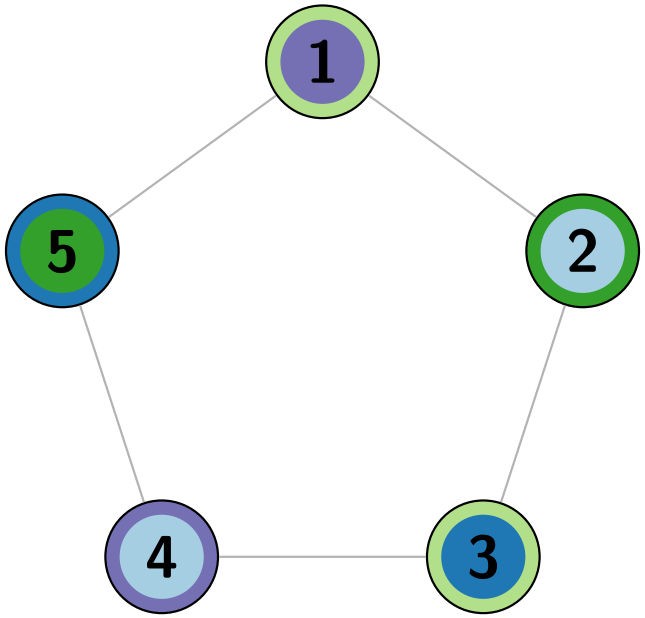
Fractional Colouring?



- If we change the rules, we only need $\frac{5}{2}$ colours.
- $\omega \leq \chi_f \leq \chi$
- Finding χ_f is NP-hard.

The Lovász Number?

- $\omega \leq \overline{\vartheta} \leq \chi$

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- And $\overline{\vartheta}$ can be found in polynomial time...
- ...in theory.

Selected Exact Maximum Clique Algorithms

- Branch and bound, using constructive greedy colouring as a bound and for ordering:
  - Tomita et. al.: MCQ (2003), MCR (2007), MCS (2010)

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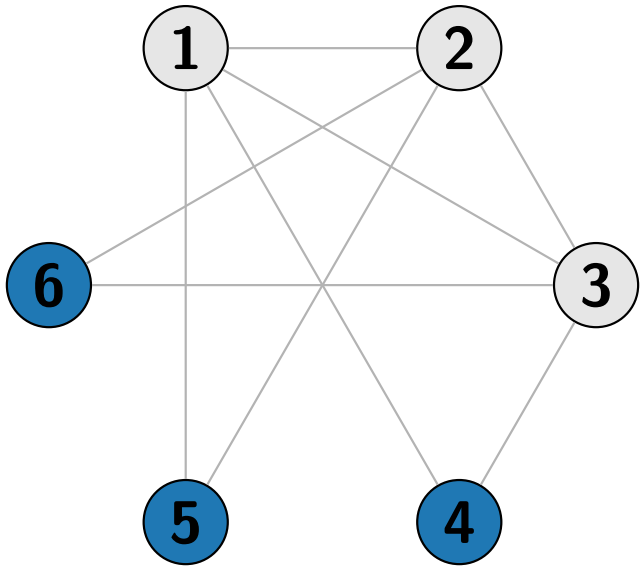
  - Li and Quan: MaxCLQ (2010)

Independent Sets

Maximum Independent Set Algorithms



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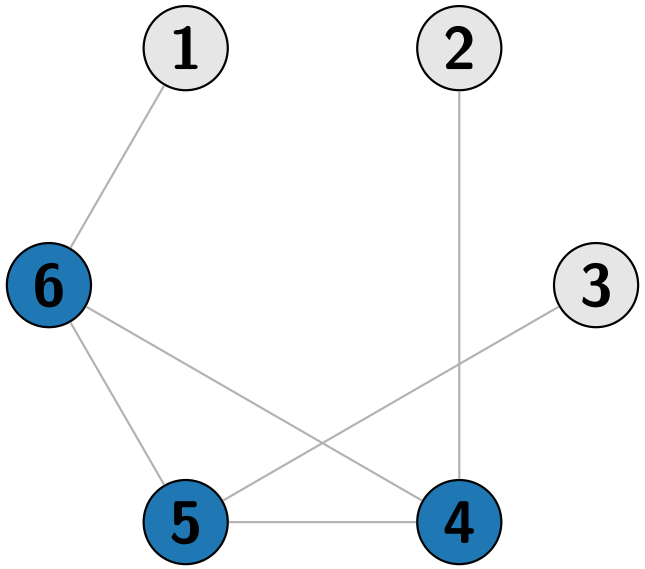
Maximum Independent Set Algorithms



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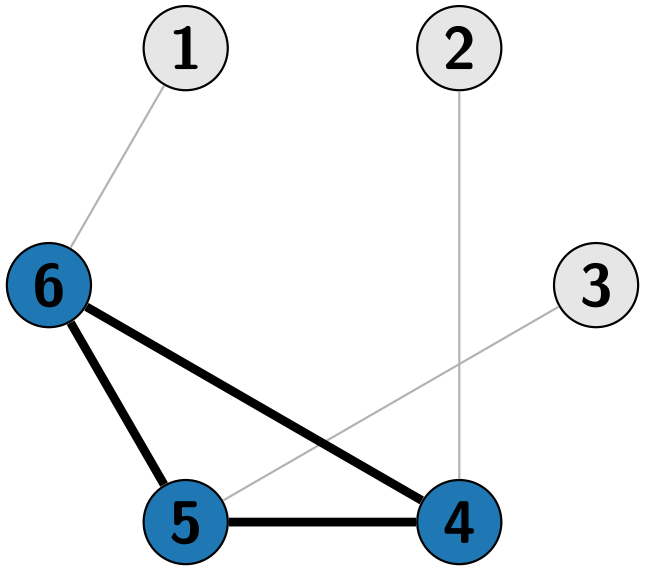
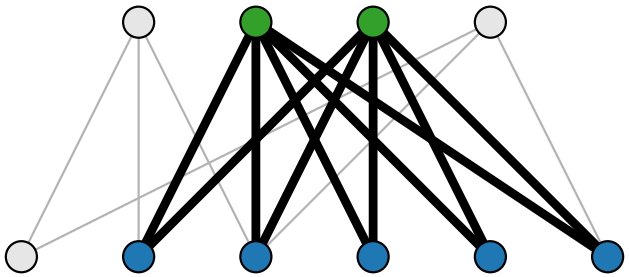
Maximum Independent Set Algorithms



- The size of a maximum independent set is denoted α. Finding α is NP-hard.
- A clique in $\overline{G}$ is an independent set in G, so we may reuse our maximum clique algorithms.
- A colouring of $\overline{G}$ is a *clique cover* of G.

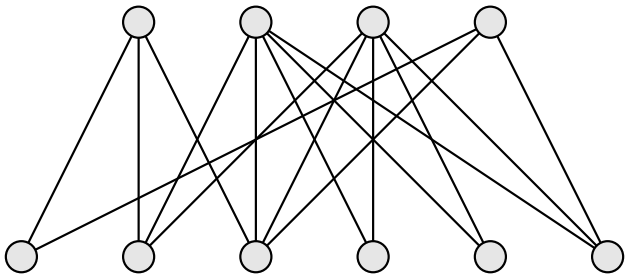
Bicliques

Bicliques in Bipartite Graphs



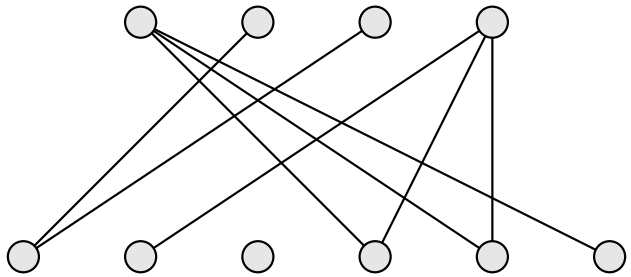
- An *induced biclique* in a graph is two independent sets A and B, and with every vertex in A adjacent to every vertex in B and vice-versa.

Bicliques in Bipartite Graphs



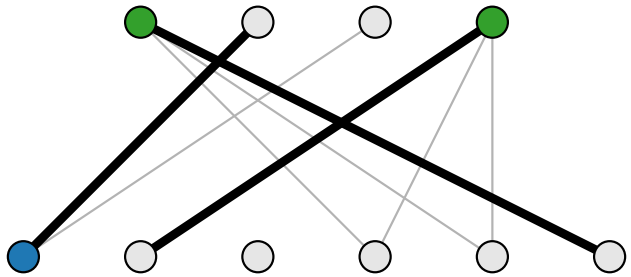
- We can find a maximum vertex biclique in a bipartite graph in polynomial time.

Bicliques in Bipartite Graphs



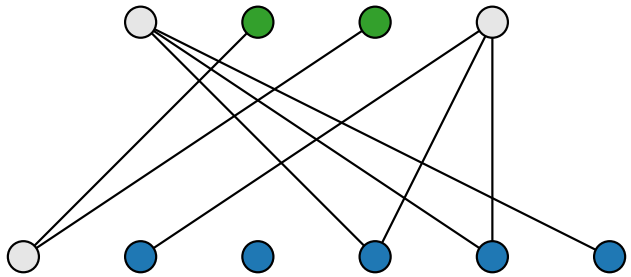
- We start by taking the bipartite complement of the graph, to get a new bipartite graph.

Bicliques in Bipartite Graphs



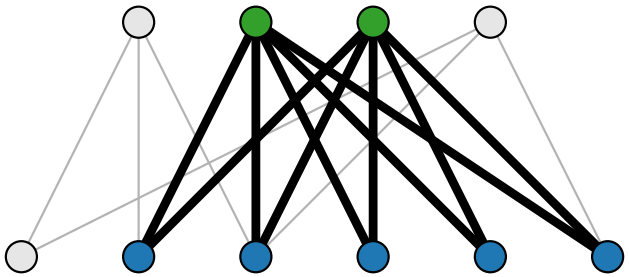
- Then we find a maximum cardinality matching, and use König's Theorem to get a minimum vertex cover.

Bicliques in Bipartite Graphs



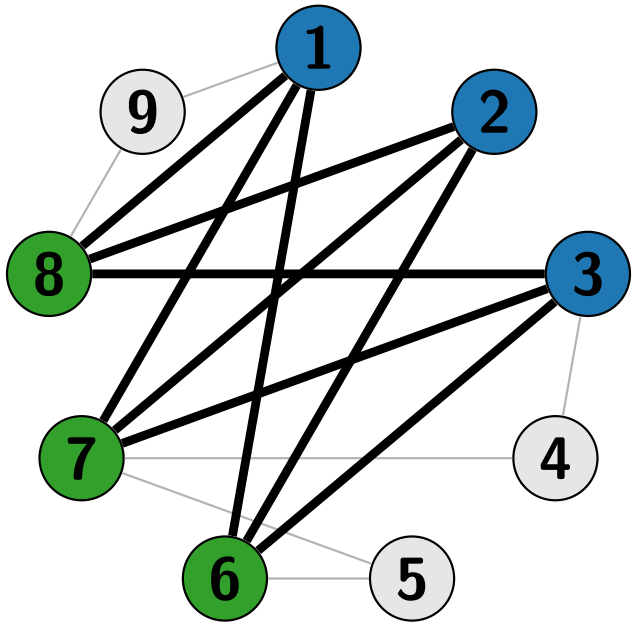
- The vertex set complement of a minimum vertex cover is a maximum independent set.

Bicliques in Bipartite Graphs



- And a maximum independent set in the bipartite complement gives us a maximum biclique in the original graph.

Balanced Induced Bicliques



- An induced biclique is *balanced* if $|A| = |B|$.
- Determining the size of a maximum balanced induced biclique, which we denote $\ddot{\omega}$, is NP-hard.

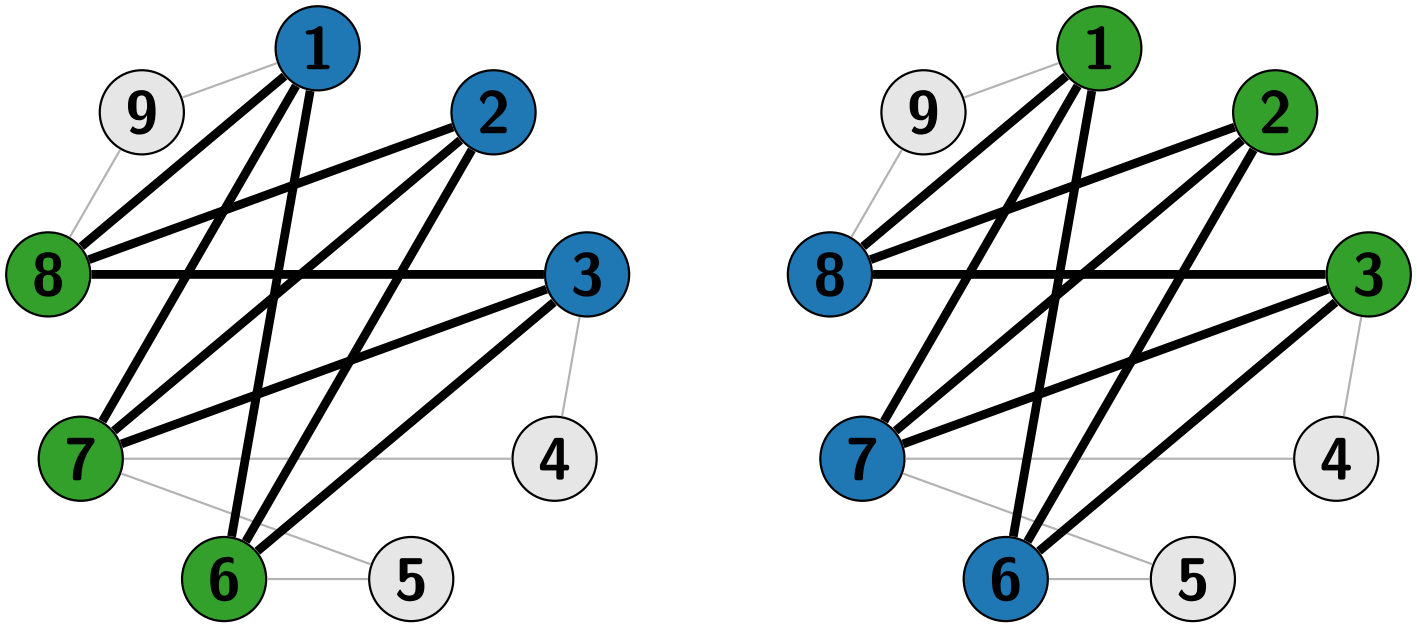
Branch and Bound for Maximum Balanced Biclique

- Grow two independent sets, adding vertices to alternating sides.
- When picking a vertex, throw away adjacent vertices on the same side, and non-adjacent vertices on the other side.

Branch and Bound for Maximum Balanced Biclique

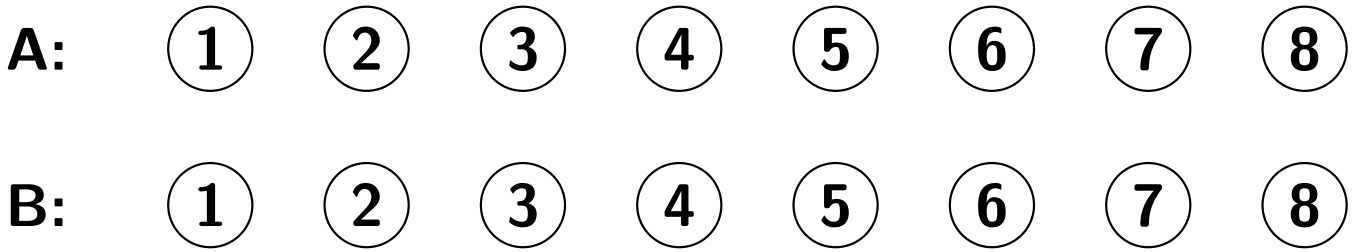
- Grow two independent sets, adding vertices to alternating sides.
- When picking a vertex, throw away adjacent vertices on the same side, and non-adjacent vertices on the other side.
- Be careful with the balance condition: the size of the two independent sets can differ by 1.

Symmetry



- We are growing an *ordered* pair of sets.
- But (A, B) is a balanced induced biclique iff (B, A) is.

Excluding Symmetries



- Use a 0-1 encoding for A and B.

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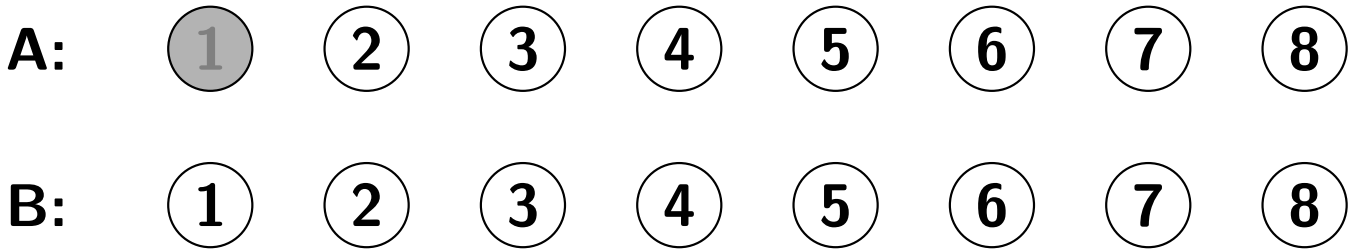

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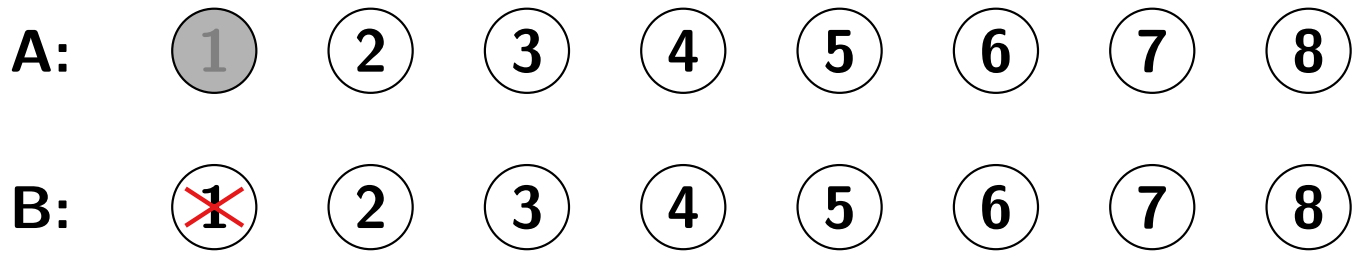

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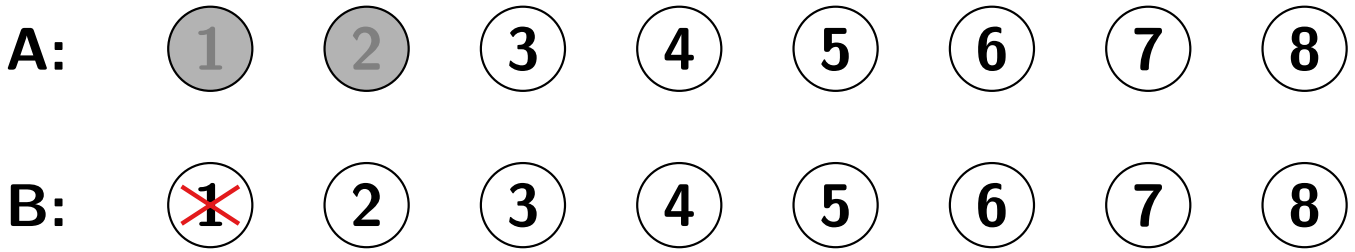

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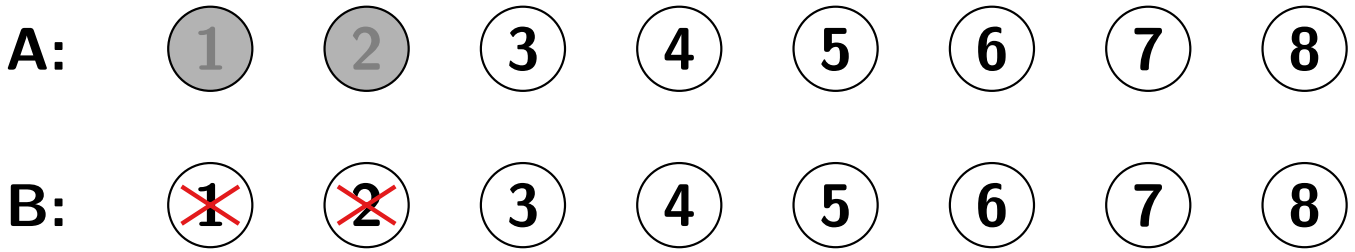

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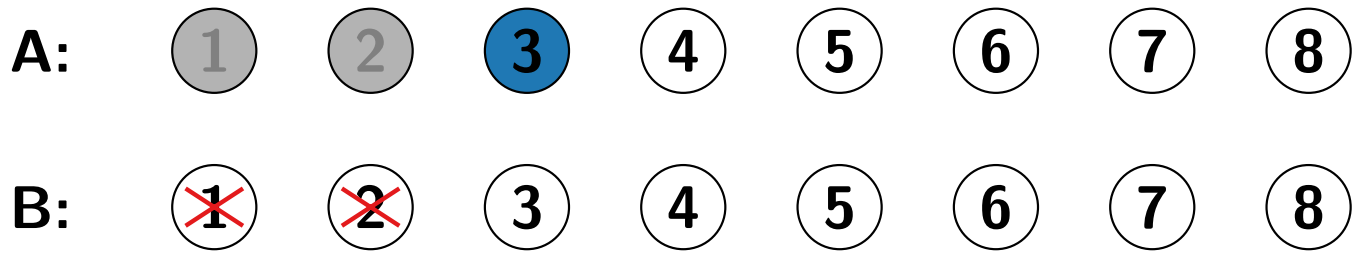

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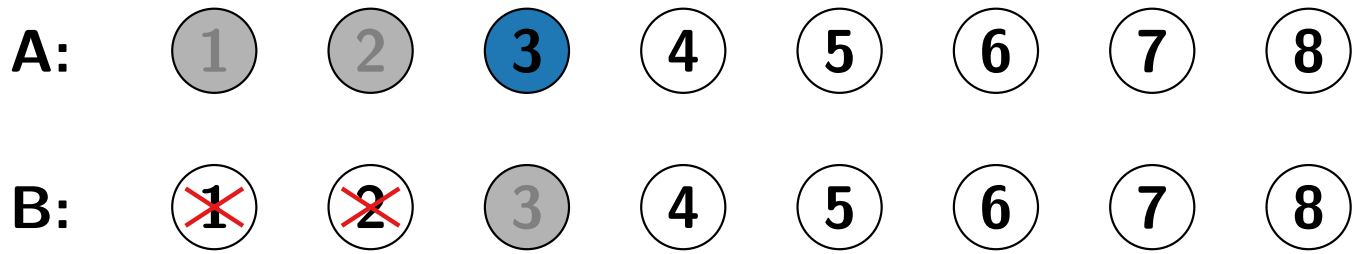

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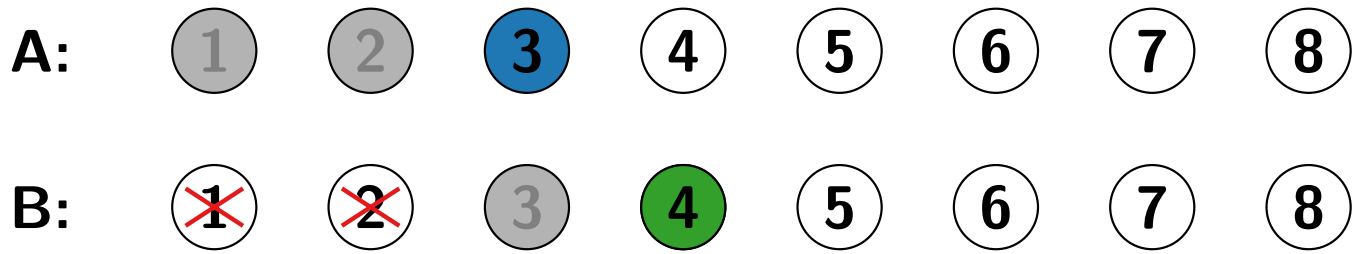

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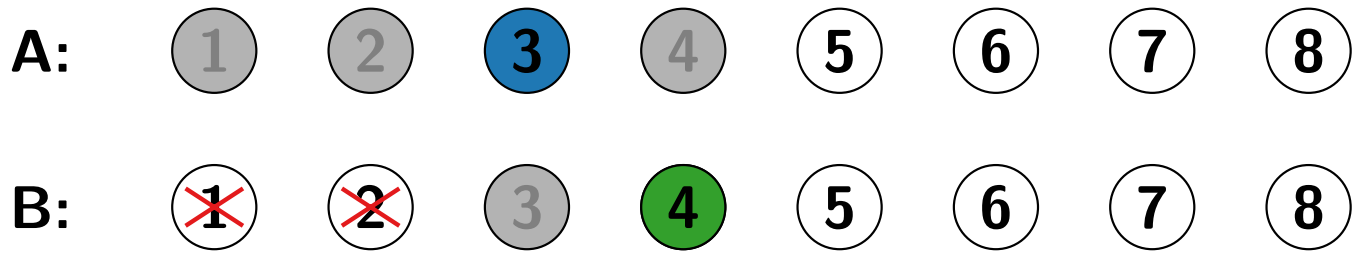

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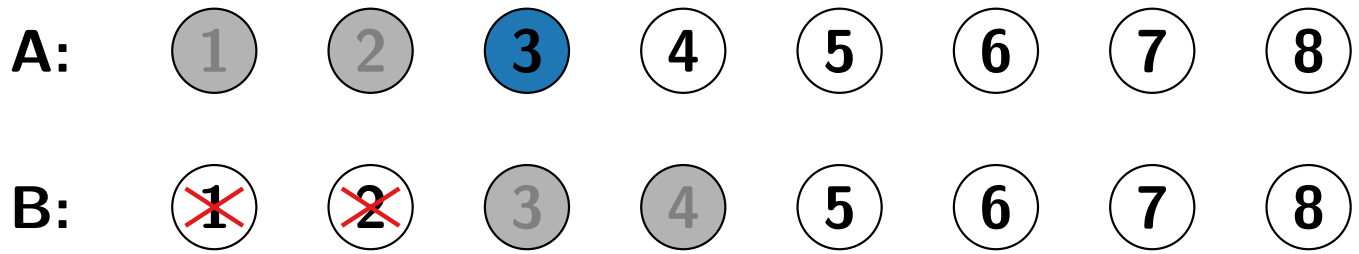

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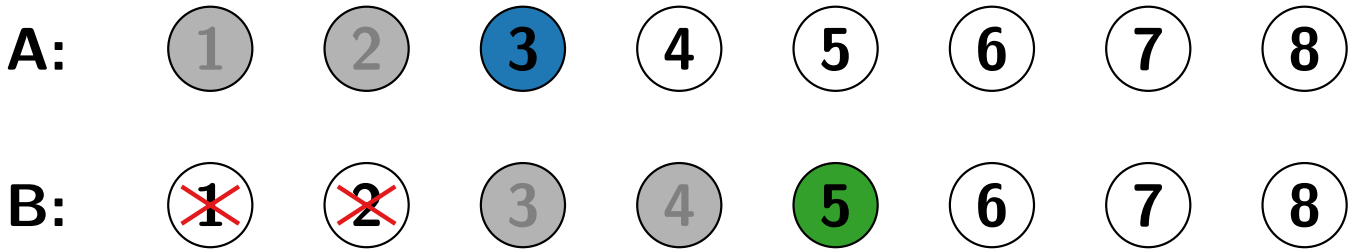

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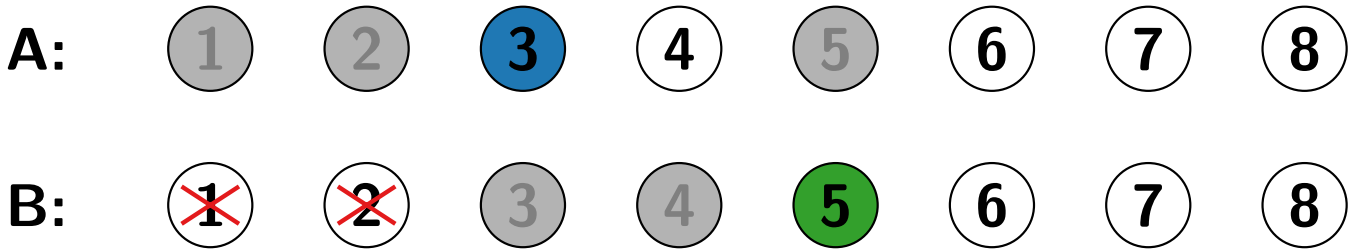

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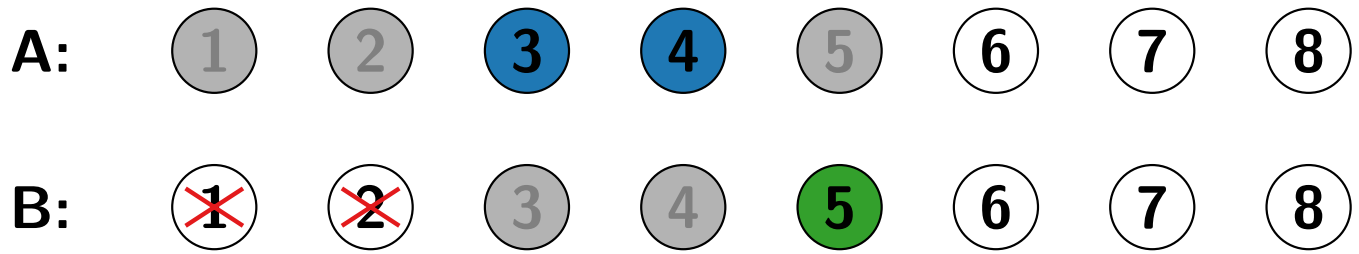

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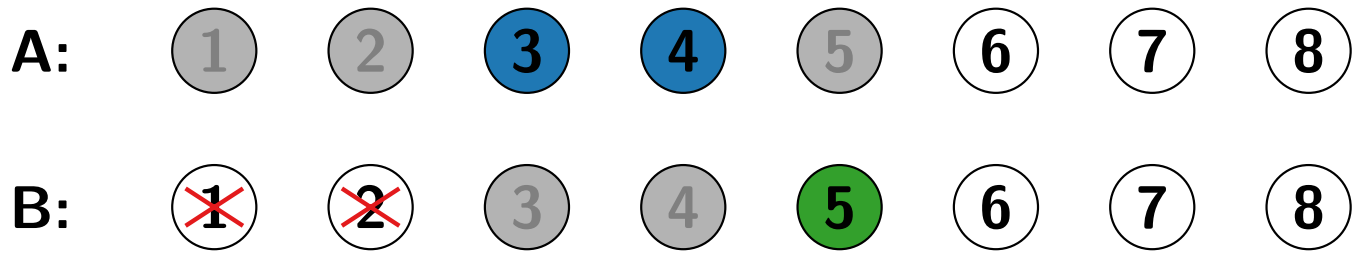

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- So we force $A > B$ lexicographically (and we can do this in decision order).

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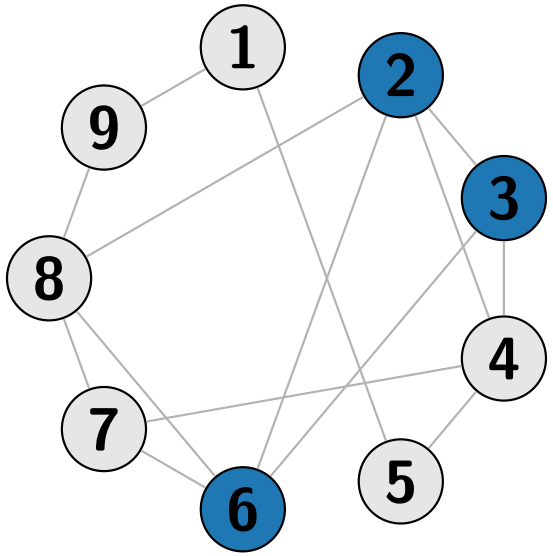
- We can detect this: a greedy clique cover of n vertices uses n cliques iff we have an independent set. We can then do something different for a bound.
  - The size of a balanced biclique is at most the size of a biclique. But the cost of calculation seems to outweigh the pruning.
  - Something involving relative degrees?
  - Indirect colouring?

k-Cliques

k-Clique



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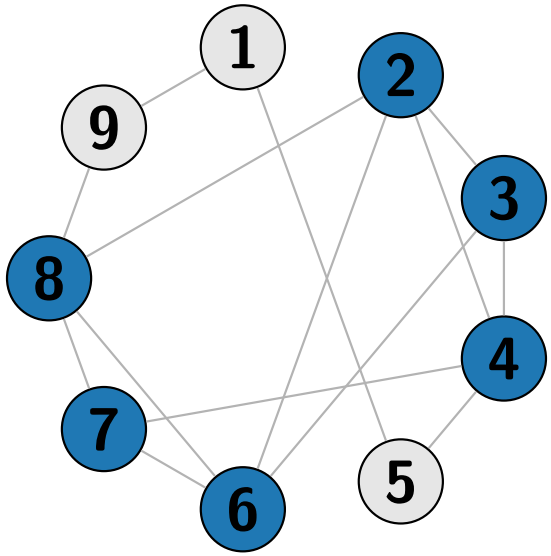
k-Clique



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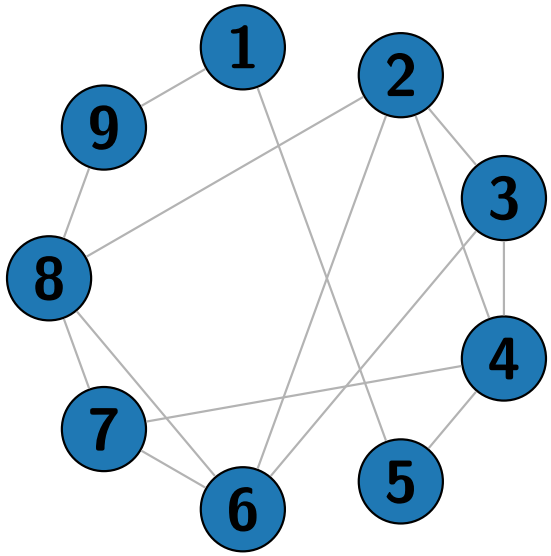

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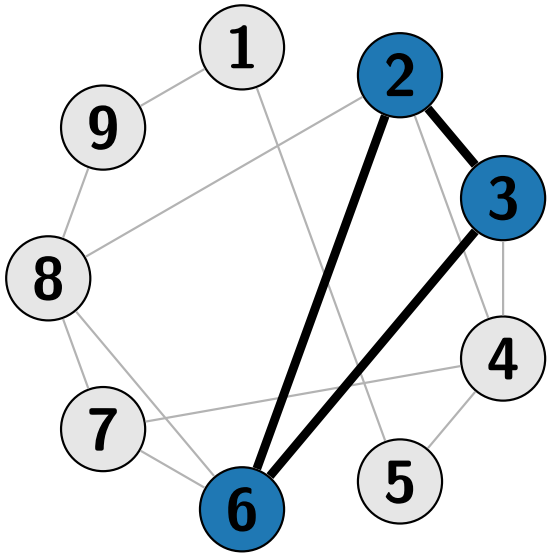
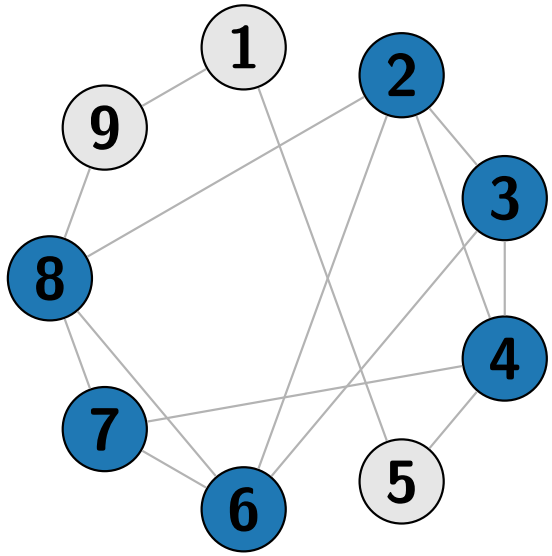
- A 2-clique is "a set of people who all either know each other, or have a mutual acquaintance".
- A k-clique is a set of vertices, each pair of which is connected by a path of length at most k.

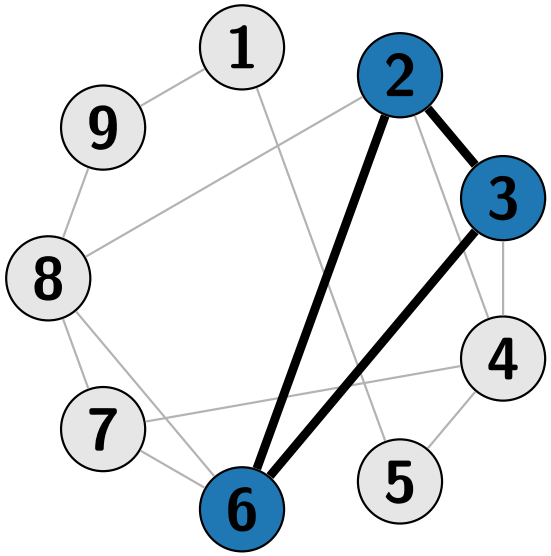
k-Clique Algorithms

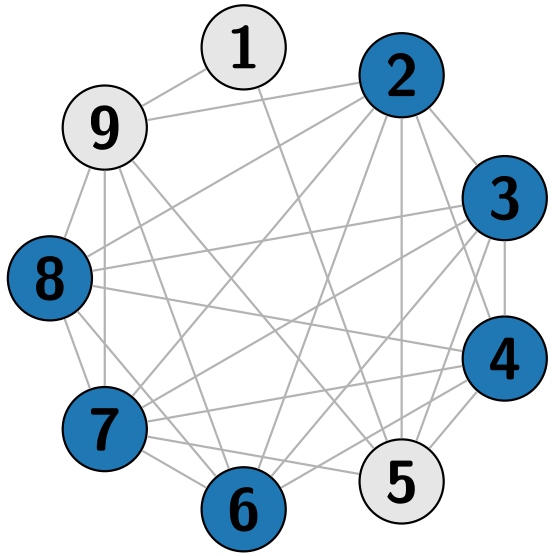




- The size of a maximum k-clique is denoted $\tilde{\omega}_k$. Finding $\tilde{\omega}_k$ is NP-hard.

k-Clique Algorithms





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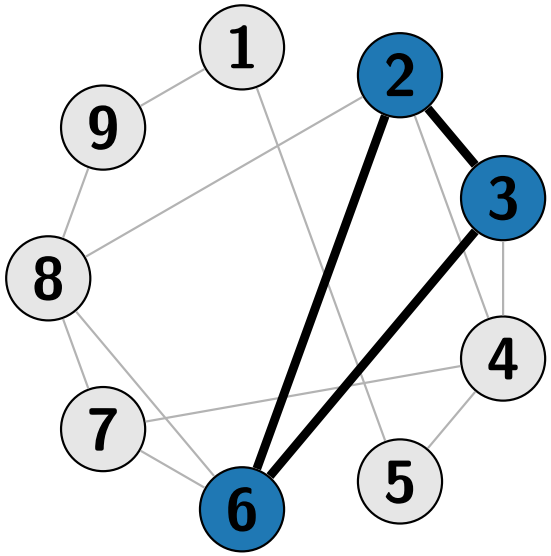
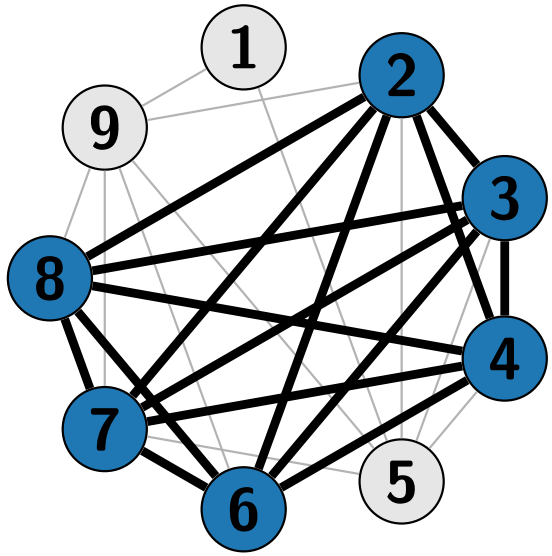
- A clique in G^k is a k-clique in G, so we may reuse our maximum clique algorithms.

k-Clique Algorithms





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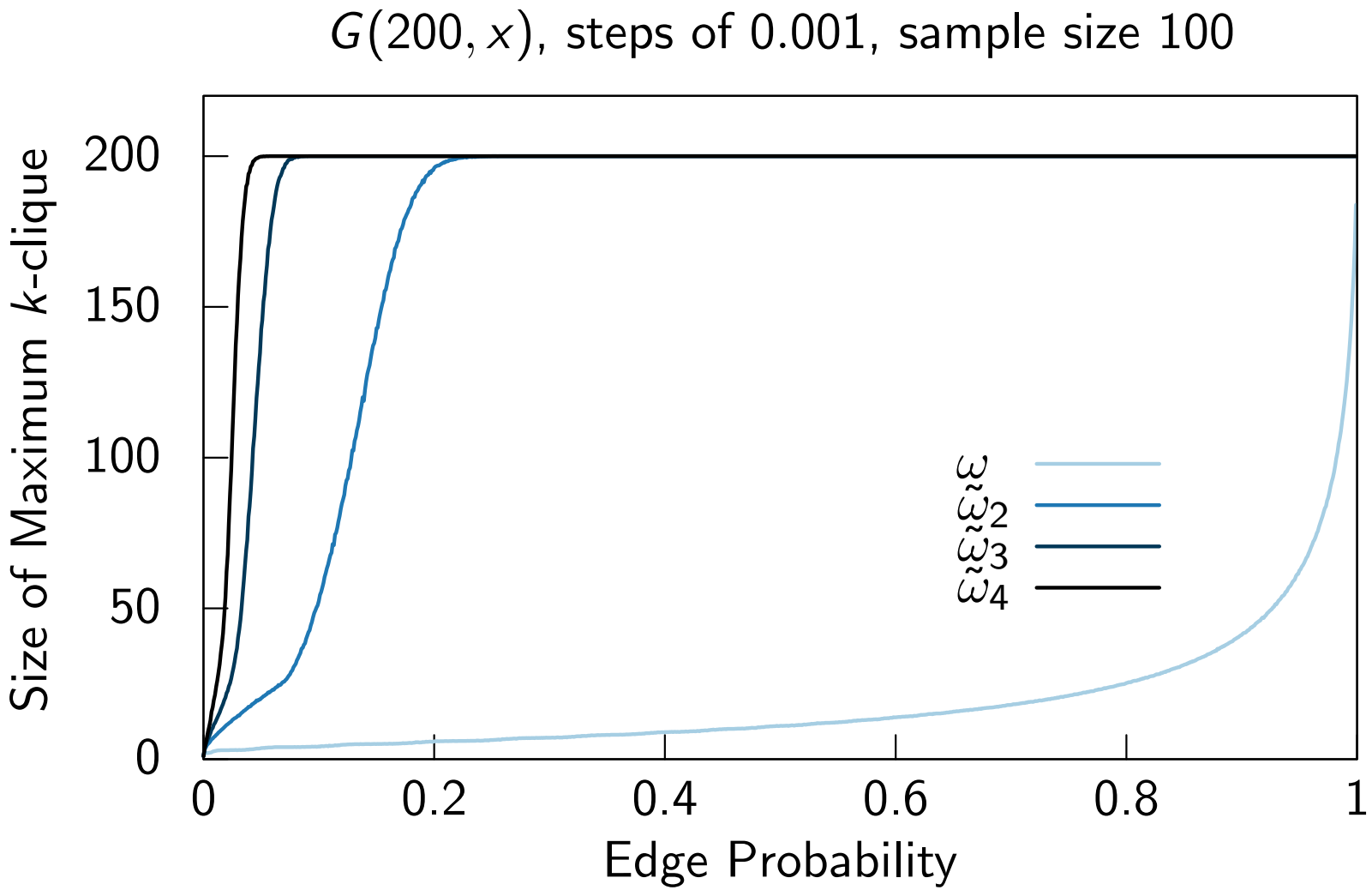
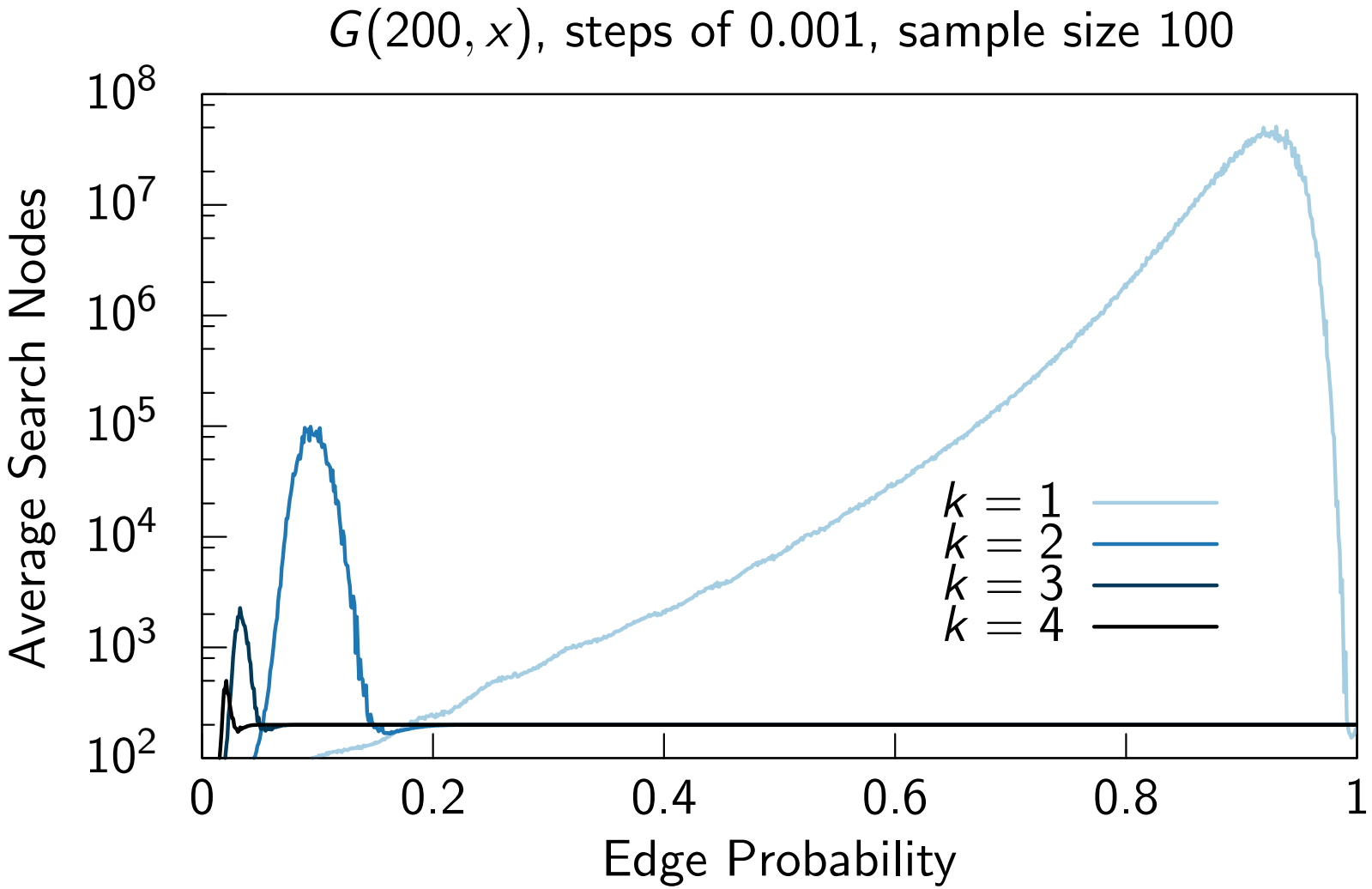
How Big is $\tilde{\omega}_k$?

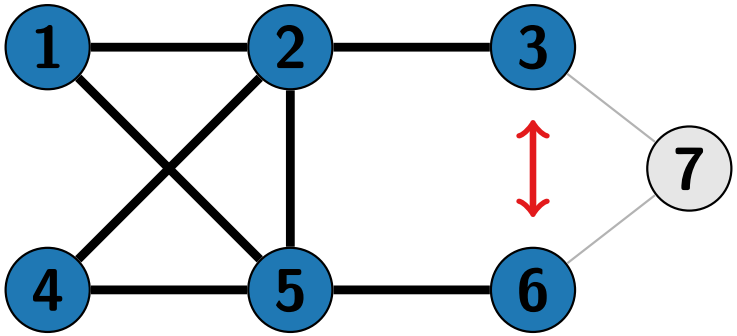


How Hard is Solving $\tilde{\omega}_k$?



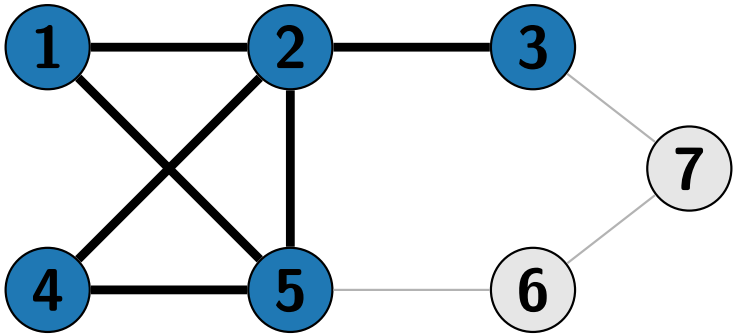
k-Clubs

k-Club



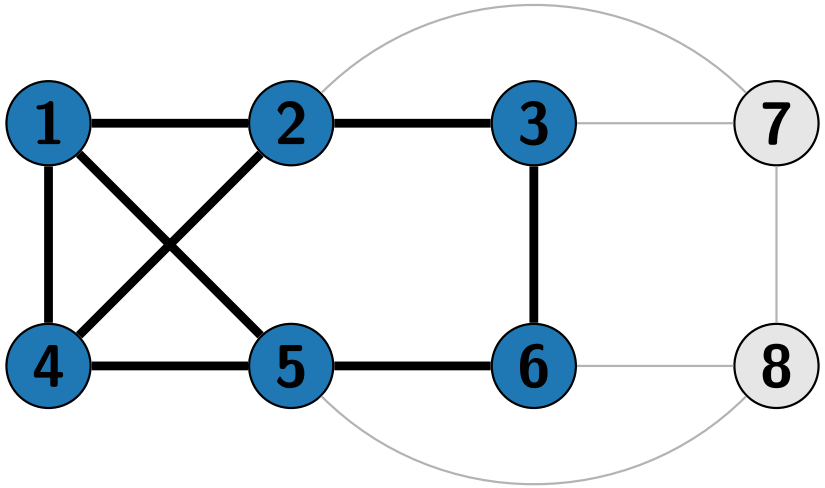
- A k-clique can use paths "that go outside the set".

k-Club



- A k-clique can use paths "that go outside the set".
- A k-club is a set of vertices, each pair of which is connected by a path of length at most k, using only vertices in this set.
- The size of a maximum k-club is denoted $\bar{\omega}_k$. Finding $\bar{\omega}_k$ is NP-hard.

k-Clubs Don't Grow on Trees



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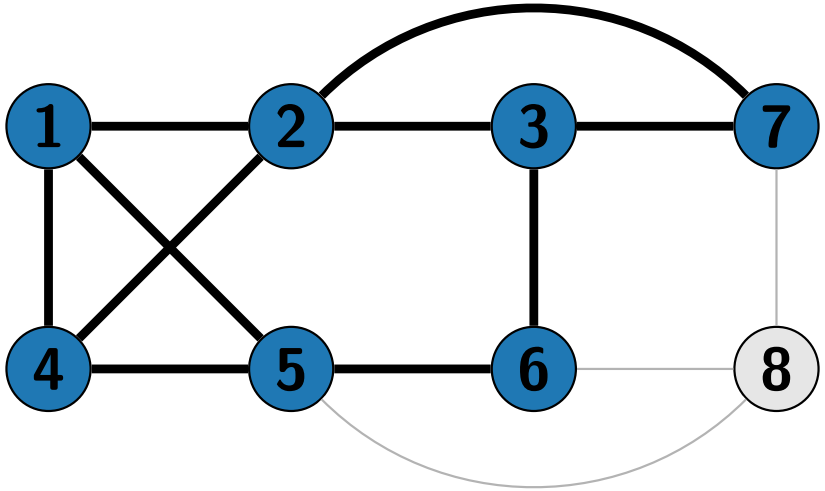

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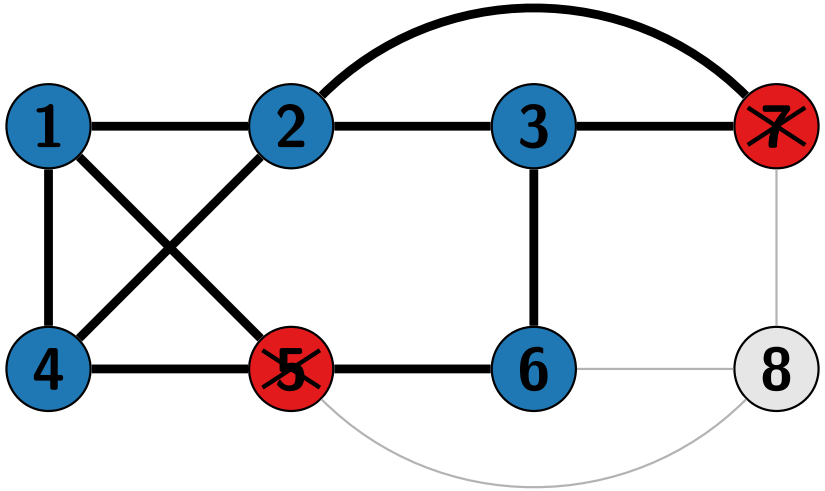

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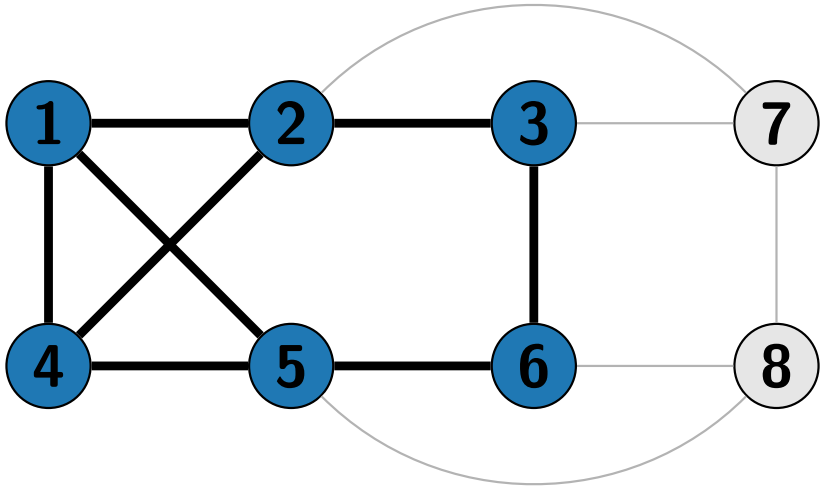

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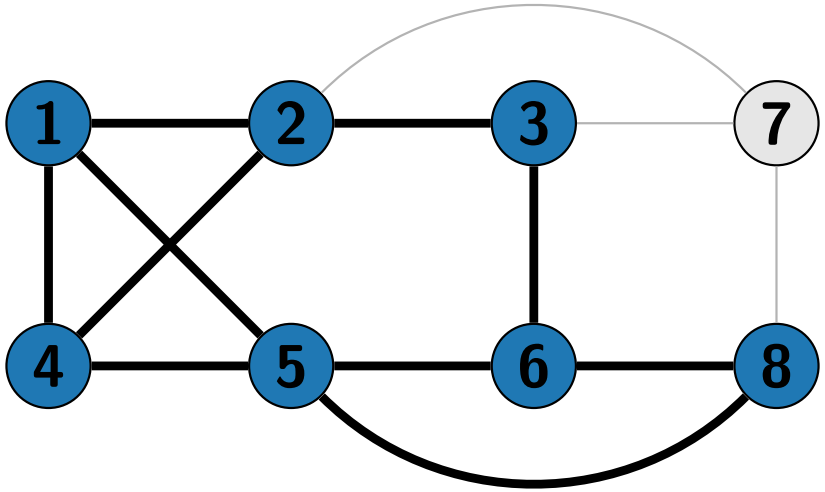

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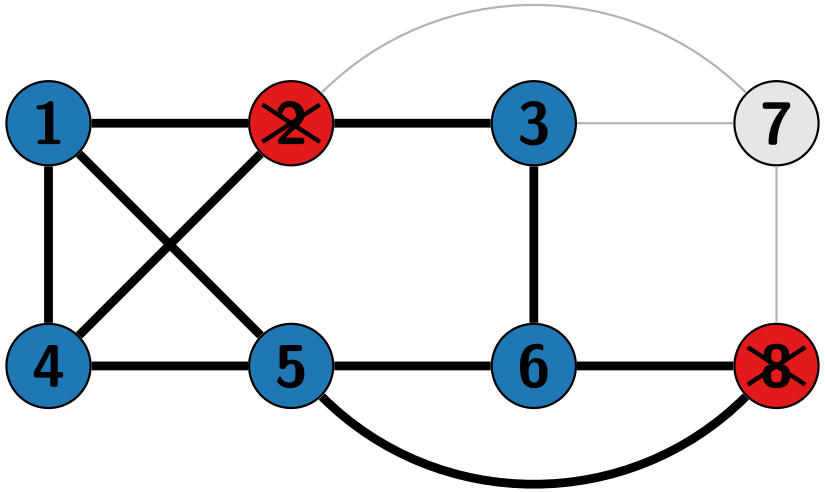

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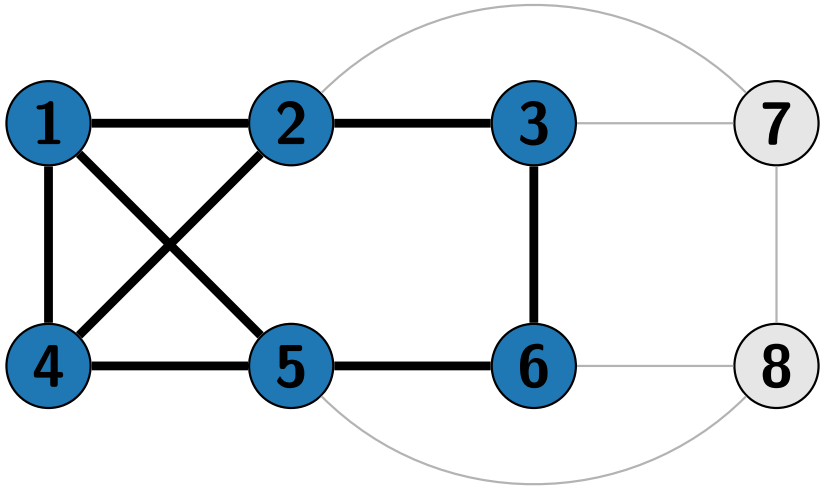

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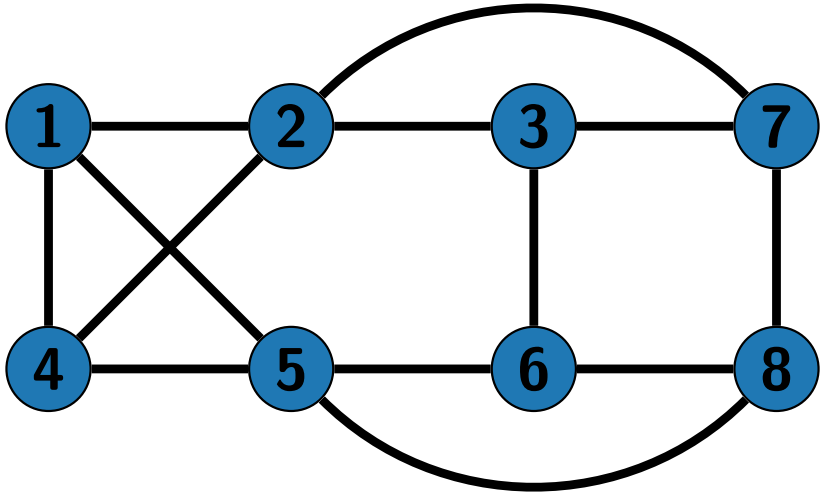

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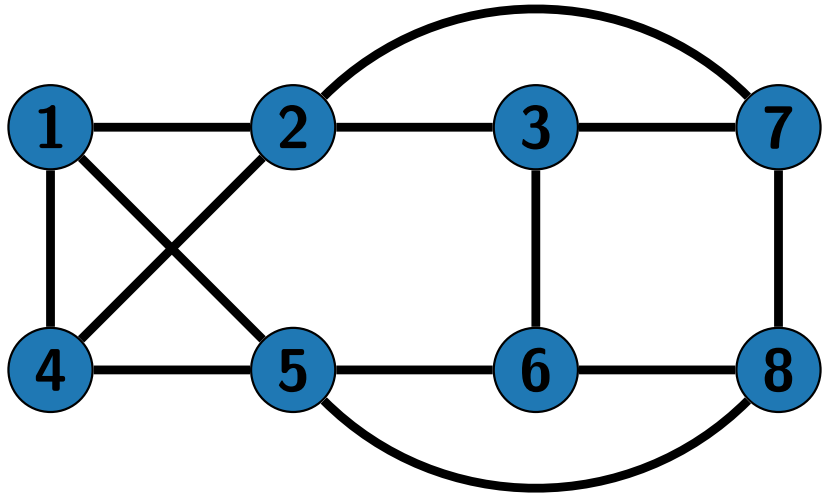
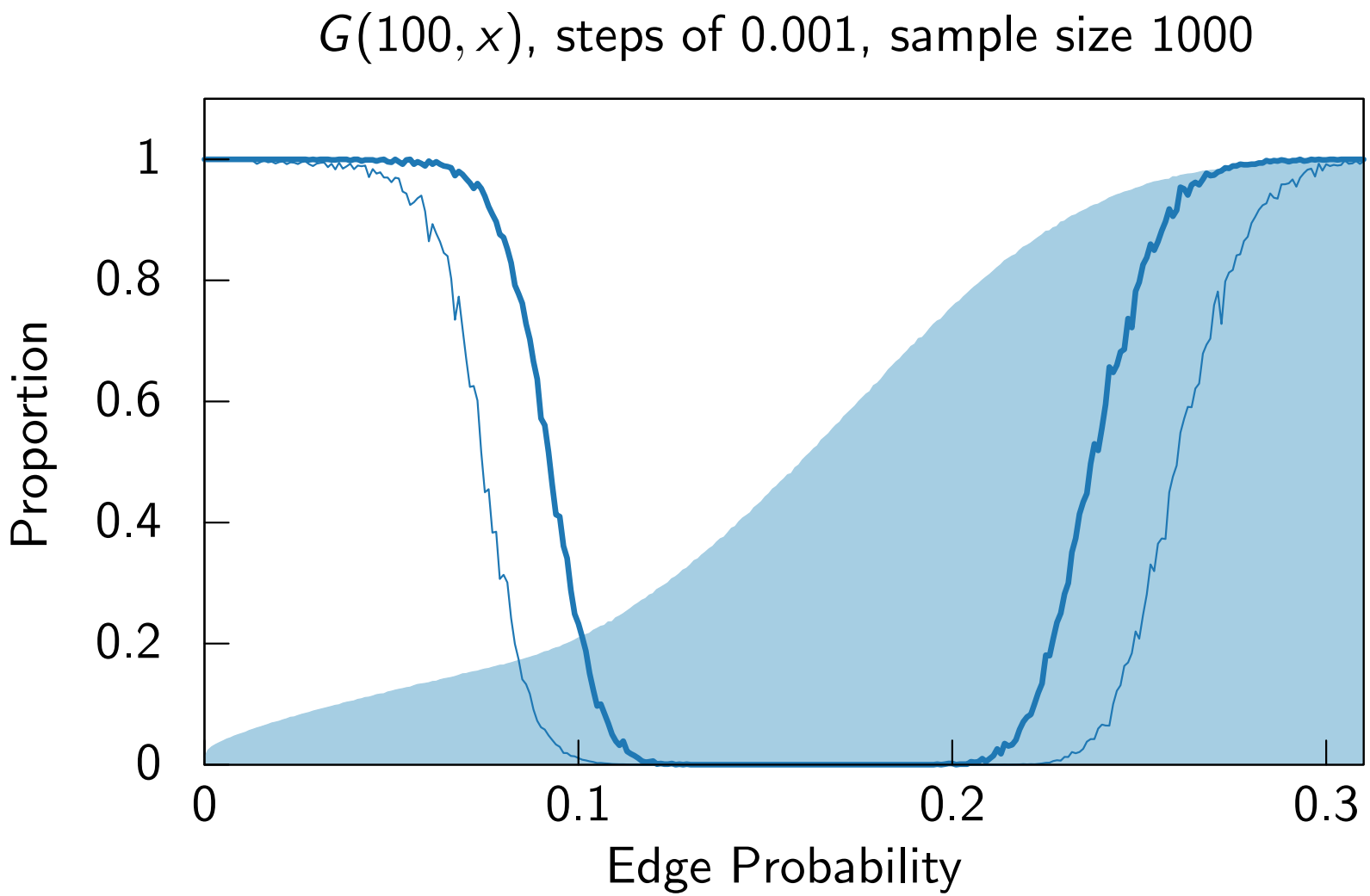


- We can't grow k-clubs recursively, one vertex at a time (being a k-club is not *hereditary on induced subgraphs*).
- Determining whether a k-club is maximal is NP-complete.
- But we can at least check whether a given k-clique is a k-club.

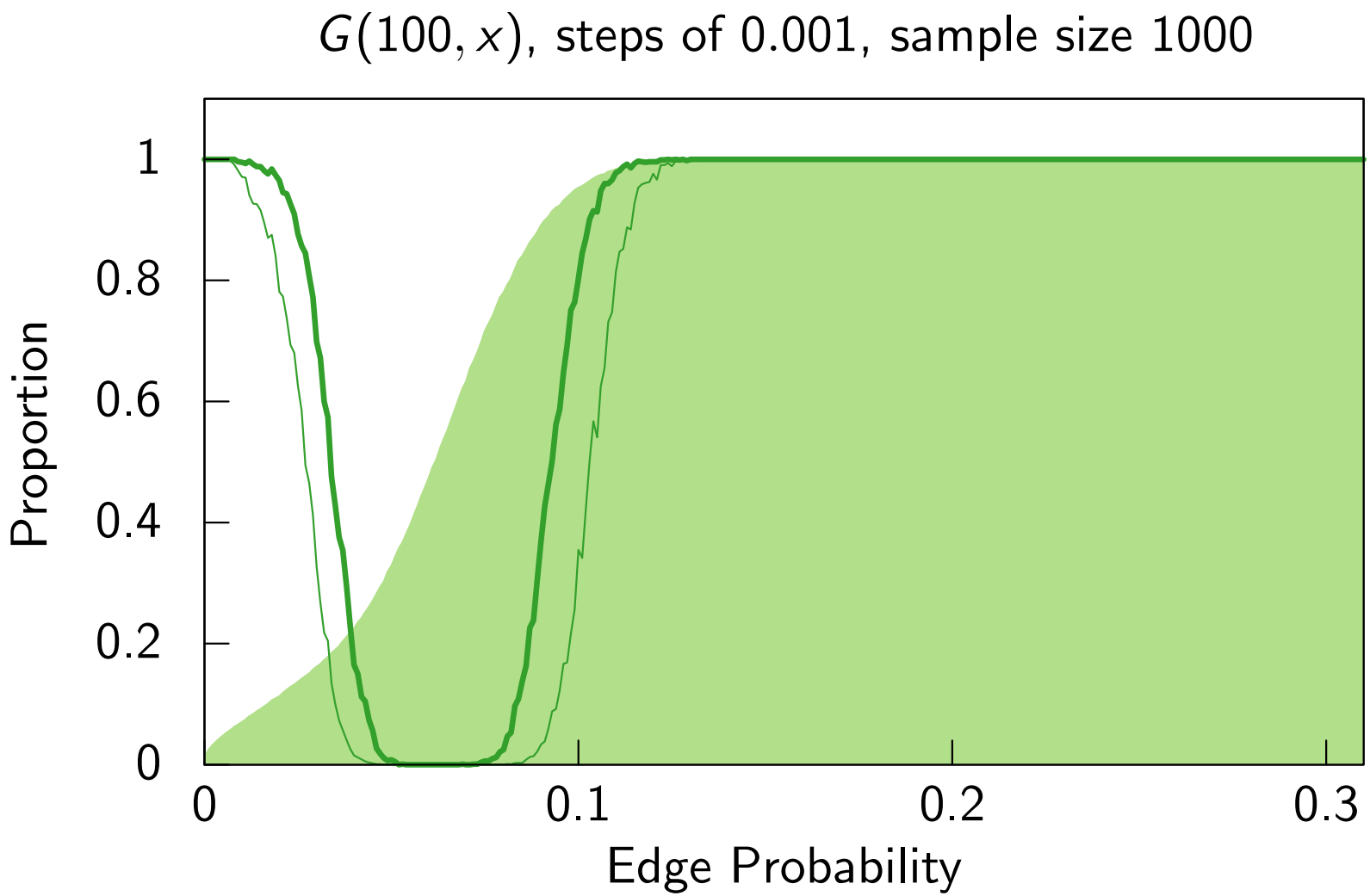
Maximum Clique Enumeration

- Find ω.
- Re-run with the incumbent fixed to $\omega - 1$.

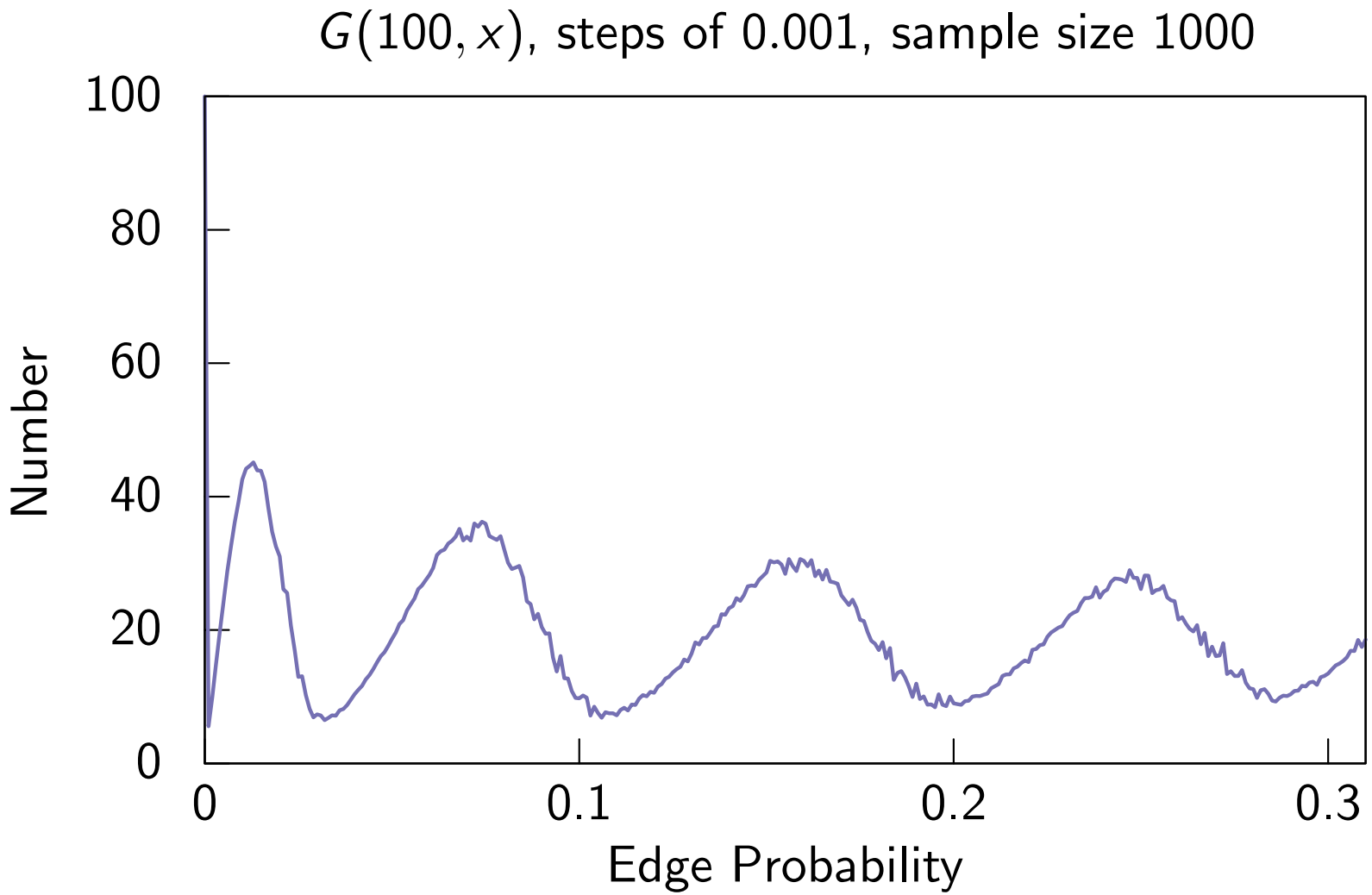
Are $\tilde{\omega}_2$ and $\bar{\omega}_2$ Different?



Are $\tilde{\omega}_3$ and $\bar{\omega}_3$ Different?



Incidentally, How Many Maximum Cliques Are There?



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