Is Your Combinatorial Search Algorithm Telling the Truth?

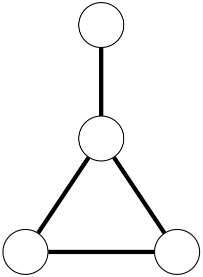
Ciaran McCreesh

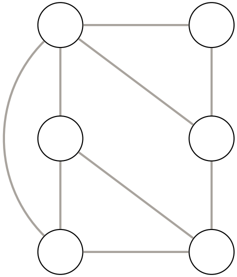
With numerous co-conspirators, including Bart Bogaerts, Jan Elffers, Stephan Gocht, Ross McBride, Matthew McIlree, Jakob Nordström, Andy Oertel, Patrick Prosser, and James Trimble

University of Glasgow

Royal Academy of Engineering

Subgraph Isomorphism





- Find the *pattern* inside the *target*.
- Applications in compilers, biochemistry, model checking, pattern recognition, …
- Often want to find *all* matches.

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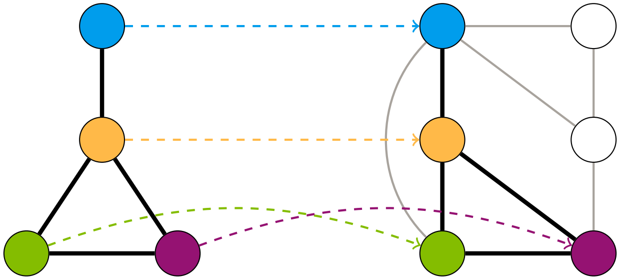

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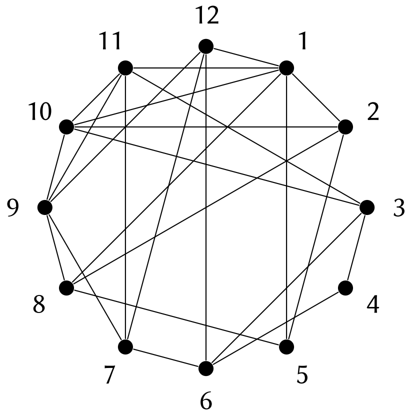
The Maximum Clique Problem



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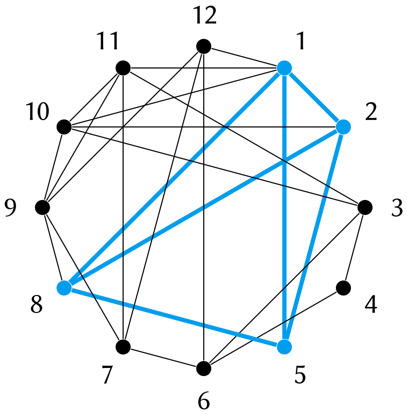

Constraint Programming

- We have a set of *variables*.
- Each variable has a finite *domain*.
- We have *constraints* between variables.
- Give each variable a value from its domain, satisfying all constraints (and maybe maximise some objective).
- Solve using inference and intelligent backtracking search.

Worst-Case Complexity vs Practice

- These problems are NP-hard, hard to approximate, etc.
- We can solve maximum clique on larger graphs than all-pairs shortest path.
- We don’t have a deep understanding as to why.

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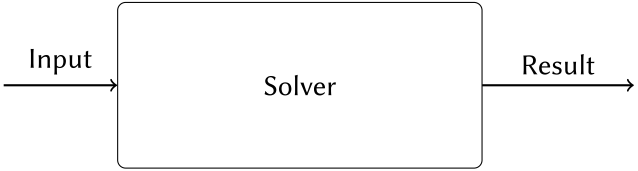
  - Empirically unsuccessful, even if people try really hard.
  - Even if you're sure, why should anyone believe you?
- Formal methods?
  - Far from being able to handle state of the art algorithms and solvers.

Proof Logging



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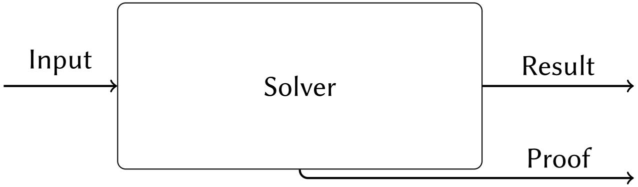

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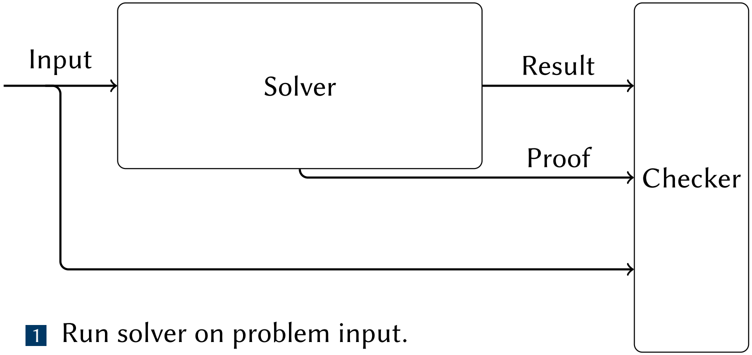

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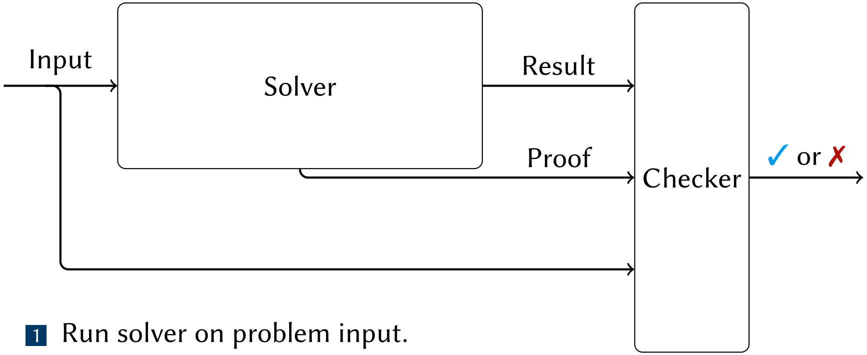

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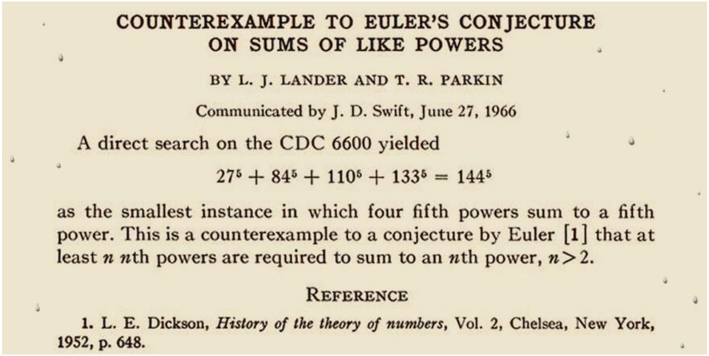
2. Get as output not only result but also proof.
3. Feed input + result + proof to proof checker.
4. Verify that proof checker says result is correct.

What Is A Proof?


COUNTEREXAMPLE TO EULER'S CONJECTURE ON SUMS OF LIKE POWERS

BY L. J. LANDER AND T. R. PARKIN

Communicated by J. D. Swift, June 27, 1966

A direct search on the CDC 6600 yielded

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

as the smallest instance in which four fifth powers sum to a fifth power. This is a counterexample to a conjecture by Euler [1] that at least n nth powers are required to sum to an nth power, $n > 2$.

REFERENCE

1. L. E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea, New York, 1952, p. 648.

The SAT Problem

- Variable x: takes value **true** (=1) or **false** (=0)
- Literal ℓ: variable x or its negation $\overline{x}$
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

The SAT Problem

Given a CNF formula F, is it satisfiable?

For instance, what about:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge$$
$$(x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

Proofs for SAT

For satisfiable instances: just specify a satisfying assignment.

For unsatisfiability: a sequence of clauses (CNF constraints).

- Each clause follows “obviously” from everything we know so far.
- Final clause is empty, meaning contradiction (written $\perp$).
- Means original formula must be inconsistent.

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Unit Propagation

Clause C unit propagates ℓ under partial assignment ρ if ρ falsifies all literals in C except ℓ.

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Proof checker should know how to unit propagate until saturation.

Davis-Putman-Logemann-Loveland (DPLL)

DPLL: Assign variables and propagate; backtrack when clause violated.

"Proof trace": when backtracking, write negation of guesses made.

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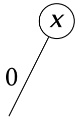

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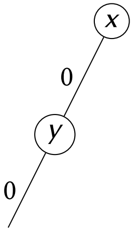

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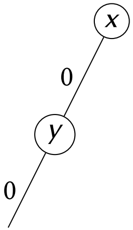

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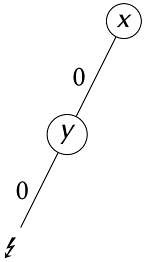

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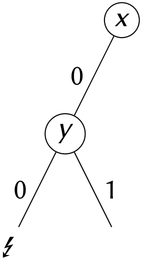

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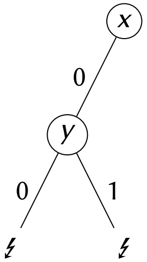

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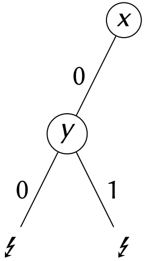
2. $x \vee \overline{y}$
3. x



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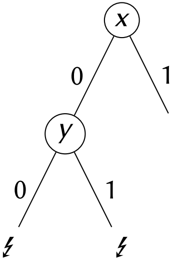

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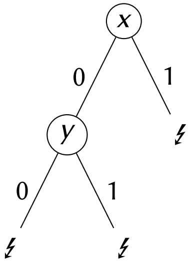

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1. $x \lor y$
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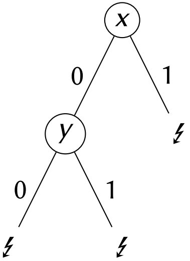
3. x
4. $\overline{x}$
5. $\bot$



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Backtrack clauses from DPLL solver generate a RUP proof.

RUP Proofs and CDCL

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is sequence of reverse unit propagation (RUP) clauses

1. $u \vee x$
2. $\overline{x}$
3. $\perp$

RUP Proofs and CDCL

Fact

All learned clauses generated by CDCL solver are RUP clauses.

So short proof of unsatisfiability for

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\cancel{\overline{x}} \vee z) \wedge (\overline{y} \vee \cancel{\overline{z}}) \wedge (\cancel{\overline{x}} \cancel{\vee} \cancel{\overline{z}}) \wedge (\overline{p} \vee \overline{u})$$

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RUP Proofs and CDCL

Fact

All learned clauses generated by CDCL solver are RUP clauses.

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$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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2. $\overline{x}$
3. $\bot$

RUP Proofs and CDCL

Fact

All learned clauses generated by CDCL solver are RUP clauses.

So short proof of unsatisfiability for

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee \cancel{x} \vee y) \wedge (\cancel{x} \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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is sequence of reverse unit propagation (RUP) clauses

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RUP Proofs and CDCL

Fact

All learned clauses generated by CDCL solver are RUP clauses.

So short proof of unsatisfiability for

$$(p \vee \cancel{\overline{u}}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee \cancel{x} \vee y) \wedge (\cancel{x} \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\cancel{\overline{p}} \cancel{\vee} \cancel{\overline{u}})$$

is sequence of reverse unit propagation (RUP) clauses

1. $u \vee x$
2. $\overline{x}$
3. $\perp$

Resolution Proofs

Fact

RUP proofs can be seen as shorthand for Resolution proofs.

Model axioms From the input

Resolution

$$\frac{x_1 \lor x_2 \lor \ldots \lor x_i \lor c \qquad \overline{c} \lor y_1 \lor y_2 \lor \ldots y_j}{x_1 \lor x_2 \lor \ldots \lor x_i \lor y_1 \lor y_2 \lor \ldots \lor y_j}$$

- To prove unsatisfiability: resolve until you reach the empty clause.

Resolution Can’t Count

- In subgraph isomorphism, can’t map a pattern vertex with n vertices into a target graph with $n - 1$ vertices.
- This requires exponential length proofs in resolution!

From CNF to Pseudo-Boolean

- A set of $\{0, 1\}$-valued variables x_i, 1 means true.
- Constraints are linear inequalities

$$\sum_i c_i x_i \geq C$$

- Write $\overline{x}_i$ to mean $1 - x_i$.
- Can rewrite CNF to pseudo-Boolean directly,

$$x_1 \vee \overline{x}_2 \vee x_3 \qquad \leftrightarrow \qquad x_1 + \overline{x}_2 + x_3 \geq 1$$

Cutting Planes Proofs

Model axioms From the input

Literal axioms
$$\frac{}{\ell_i \geq 0}$$

Addition
$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

Multiplication
for any $c \in \mathbb{N}^+$
$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

Division
for any $c \in \mathbb{N}^+$
$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

Interleaving RUP and Cutting Planes

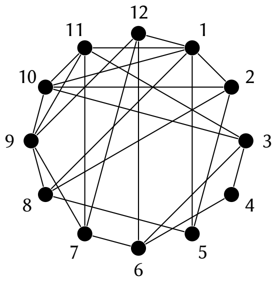
- Can define RUP similarly for pseudo-Boolean constraints.
- It does the same thing on clauses.
- Idea: use RUP for backtracking, and include explicit cutting planes steps to justify reasoning.

The VeriPB System

`https://gitlab.com/MIAOresearch/software/VeriPB`

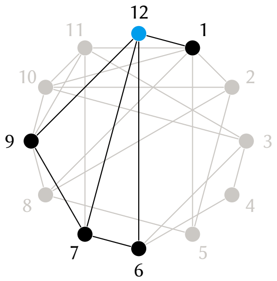
- MIT licence, written in Python with parsing in C++.
- Useful features like tracing and proof debugging.

(Roughly) How (Some) Maximum Clique Solvers Work



- Pick a vertex v, and branch on whether or not to include it. When acccepting, reject any vertices not adjacent to v.
- Remember largest clique found so far, and only look for bigger cliques.

(Roughly) How (Some) Maximum Clique Solvers Work



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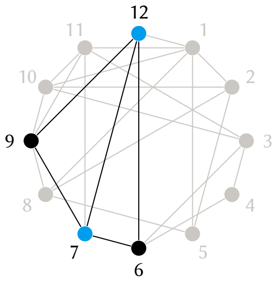

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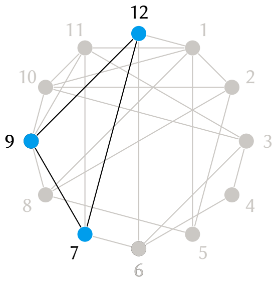
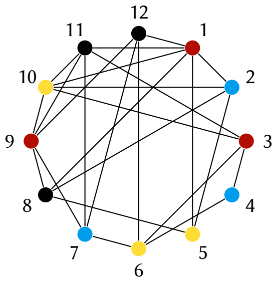


- Pick a vertex v, and branch on whether or not to include it. When acccepting, reject any vertices not adjacent to v.
- Remember largest clique found so far, and only look for bigger cliques.

(Roughly) How (Some) Maximum Clique Solvers Work



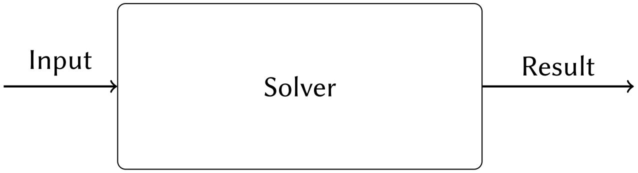
Given a *k*-colouring of a subgraph, that subgraph cannot have a clique of more than *k* vertices.

- Each colour class describes an at-most-one constraint.

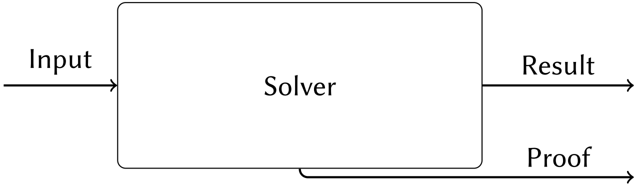
Making a Proof-Logging Clique Solver

1. Output a pseudo-Boolean encoding of the problem.
   - Clique problems have several standard file formats.
2. Make the solver log its search tree.
   - Output a small header.
   - Output something on every backtrack.
   - Output something every time a solution is found.
   - Output a small footer.
3. Figure out how to log the bound function.

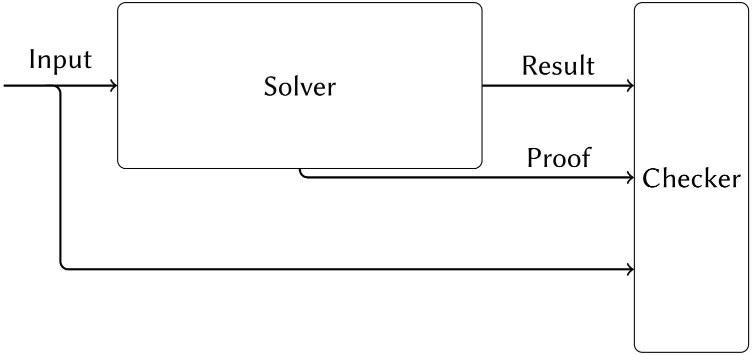
A Slightly Different Workflow



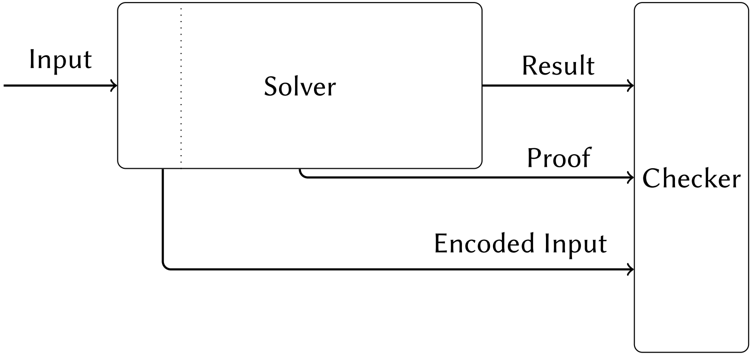
A Slightly Different Workflow



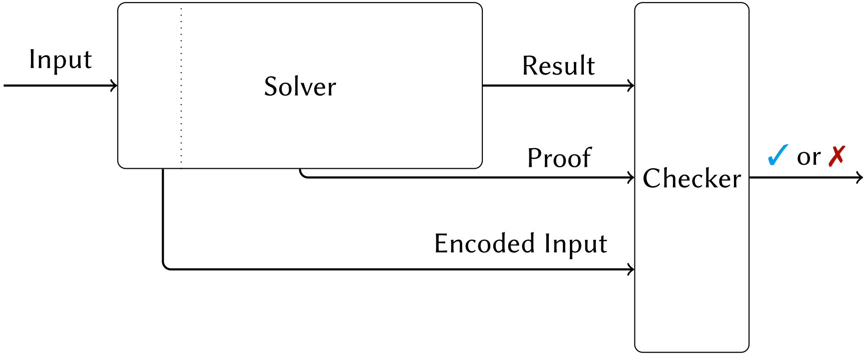
A Slightly Different Workflow



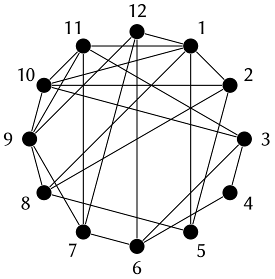
A Slightly Different Workflow



A Slightly Different Workflow



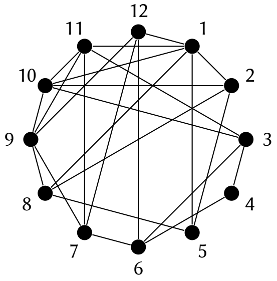
A Pseudo-Boolean Encoding for Clique (in OPB Format)



```
* #variable= 12 #constraint= 41
min: -1 x1 -1 x2 -1 x3 -1 x4 . . . and so on. . .  -1 x11 -1 x12 ;
1 ~x3 1 ~x1 >= 1 ;
1 ~x3 1 ~x2 >= 1 ;
1 ~x4 1 ~x1 >= 1 ;
* . . . and a further 38 similar lines for the remaining non-edges
```

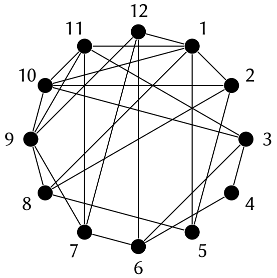
First Attempt at a Proof

```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



First Attempt at a Proof

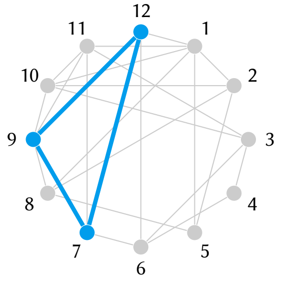
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Start with a header.
Load the 41 problem axioms.

First Attempt at a Proof

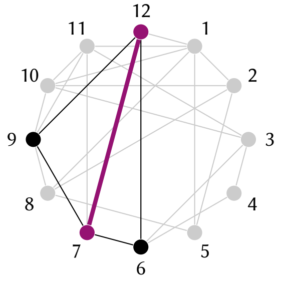
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Branch on 12, 7, 9.
Find a new incumbent.
Now looking for a ≥ 4 vertex clique.

First Attempt at a Proof

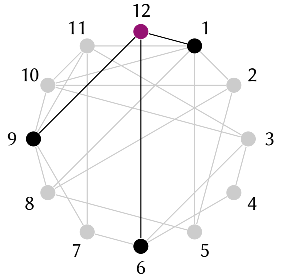
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Backtrack from 12, 7.
Only 6 and 9 feasible.
No ≥ 4 vertex clique possible.
Effectively this deletes the 7–12 edge.

First Attempt at a Proof

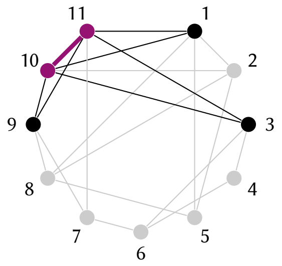
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Backtrack from 12.
Only 1, 6 and 9 feasible.
No ≥ 4 vertex clique possible.
Effectively this deletes vertex 12.

First Attempt at a Proof

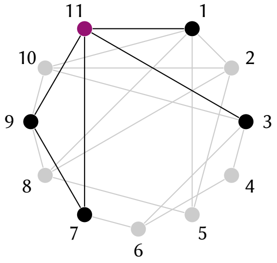
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Branch on 11 then 10.
Only 1, 3 and 9 feasible.
No ≥ 4 vertex clique possible.
Backtrack, deleting the edge.

First Attempt at a Proof

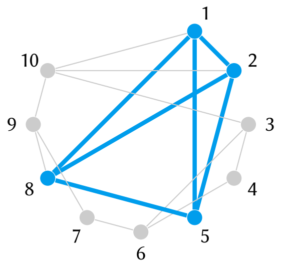
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Backtrack from 11.
Clearly no ≥ 4 clique.
Delete the vertex.

First Attempt at a Proof

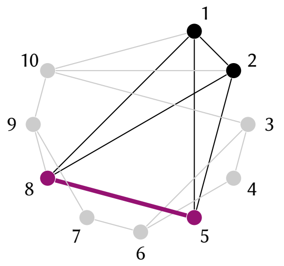
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Branch on 8, 5, 1, 2.
Find a new incumbent.
Now looking for a ≥ 5 vertex clique.

First Attempt at a Proof

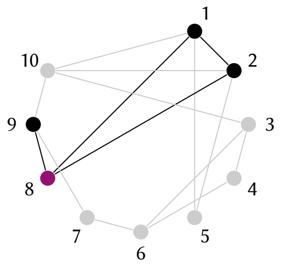
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Backtrack from 8, 5.
Only 4 vertices, can't have a ≥ 5 clique.
Delete the edge.

First Attempt at a Proof

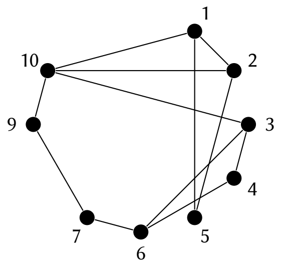
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Backtrack from 8.
Still not enough vertices.
Delete the vertex.

First Attempt at a Proof

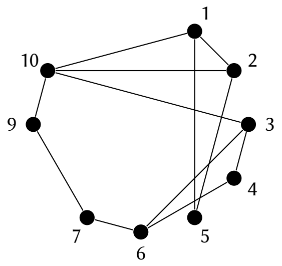
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Now obvious to solver that claim of ≥ 5 clique is contradictory (remaining vertices are 3-colourable).

First Attempt at a Proof

```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Assert previous line has derived contradiction, ending proof.

Verifying This Proof (Or Not. . .)

```
$ veripb clique.opb clique-attempt-one.veripb
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```

Verifying This Proof (Or Not. . .)

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$ veripb clique.opb clique-attempt-one.veripb
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
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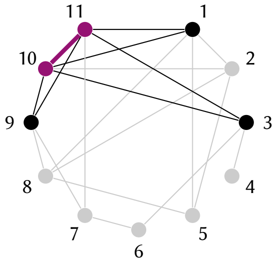

Verifying This Proof (Or Not…)

```
$ veripb --trace clique.opb clique-attempt-one.veripb
line 002: f 41
  ConstraintId 001: 1 ~x1 1 ~x3 >= 1
  ConstraintId 002: 1 ~x2 1 ~x3 >= 1
...
  ConstraintId 041: 1 ~x11 1 ~x12 >= 1
line 003: o x7 x9 x12 ~x1 ~x2 ~x3 ~x4 ~x5 ~x6 ~x8 ~x10 ~x11
  ConstraintId 042: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x7 1 x8 1 x9 1 x10 1 x1
line 004: u 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 043: 1 ~x7 1 ~x12 >= 1
line 005: u 1 ~x12 >= 1 ;
  ConstraintId 044: 1 ~x12 >= 1
line 006: u 1 ~x11 1 ~x10 >= 1 ;
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```

Recovering At-Most-One Constraints

At-most-one constraints do not follow from reverse unit propagation.

Infeasible to list every colour class we *might* use in the input.

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But we can use cutting planes to recover colour classes lazily!

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But we can use cutting planes to recover colour classes lazily!

$$\begin{aligned}
&(\overline{x}_1 + \overline{x}_6 \geq 1) && \\
+\,&(\overline{x}_1 + \overline{x}_9 \geq 1) && = 2\overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
+\,&(\overline{x}_6 + \overline{x}_9 \geq 1) && = 2\overline{x}_1 + 2\overline{x}_6 + 2\overline{x}_9 \geq 3 \\
&/\,2 && = \overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
& && \text{i.e. } x_1 + x_6 + x_9 \leq 1
\end{aligned}$$

Recovering At-Most-One Constraints

At-most-one constraints do not follow from reverse unit propagation.

Infeasible to list every colour class we *might* use in the input.

But we can use cutting planes to recover colour classes lazily!

$$\begin{aligned}
&(\overline{x}_1 + \overline{x}_6 \geq 1) && \\
+&(\overline{x}_1 + \overline{x}_9 \geq 1) && = 2\overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
+&(\overline{x}_6 + \overline{x}_9 \geq 1) && = 2\overline{x}_1 + 2\overline{x}_6 + 2\overline{x}_9 \geq 3 \\
&/\,2 && = \overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
& && \text{i.e. } x_1 + x_6 + x_9 \leq 1
\end{aligned}$$

This generalises for arbitrarily large colour classes.

- Each non-edge is used exactly once, $v(v-1)$ additions.
- $v-3$ multiplications and $v-2$ divisions.

Solvers don't need to "understand" cutting planes to write this out.

What This Looks Like

```
pseudo-Boolean proof version 1.2
f 41
o x12 x7 x9
u 1 ~x12 1 ~x7 >= 1 ;
* bound, colour classes [ x1 x6 x9 ]
p 7_{1≁6} 19_{1≁9} + 24_{6≁9} + 2 d
p 42_{obj} -1 +
u 1 ~x12 >= 1 ;
* bound, colour classes [ x1 x3 x9 ]
p 1_{1≁3} 19_{1≁9} + 21_{3≁9} + 2 d
p 42_{obj} -1 +
u 1 ~x11 1 ~x10 >= 1 ;
* bound, colour classes [ x1 x3 x7 ] [ x9 ]
p 1_{1≁3} 10_{1≁7} + 12_{3≁7} + 2 d
p 42_{obj} -1 +
u 1 ~x11 >= 1 ;
o x8 x5 x2 x1
u 1 ~x8 1 ~x5 >= 1 ;
* bound, colour classes [ x1 x9 ] [ x2 ]
p 53_{obj} 19_{1≁9} +
u 1 ~x8 >= 1 ;
* bound, colour classes [ x1 x3 x7 ] [ x2 x4 x9 ] [ x5 x6 x10 ]
p 1_{1≁3} 10_{1≁7} + 12_{3≁7} + 2 d
p 53_{obj} -1 +
p 4_{2≁4} 20_{2≁9} + 22_{4≁9} + 2 d
p 53_{obj} -3 + -1 +
p 9_{5≁6} 26_{5≁10} + 27_{6≁10} + 2 d
p 53_{obj} -5 + -3 + -1 +
u >= 1 ;
c -1
```

Verifying This Proof (For Real, This Time)

```
$ veripb --trace clique.opb clique-attempt-two.veripb
=== begin trace ===
line 002: f 41
  ConstraintId 001: 1 ~x1 1 ~x3 >= 1
  ConstraintId 002: 1 ~x2 1 ~x3 >= 1
...
  ConstraintId 041: 1 ~x11 1 ~x12 >= 1
line 003: o x7 x9 x12 ~x1 ~x2 ~x3 ~x4 ~x5 ~x6 ~x8 ~x10 ~x11
  ConstraintId 042: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x7 1 x8 1 x9 1 x10 1 x11 1 x12 >= 4
line 004: u 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 043: 1 ~x7 1 ~x12 >= 1
line 005: * bound, colour classes [ x1 x6 x9 ]
line 006: p 7 19 + 24 + 2 d
  ConstraintId 044: 1 ~x1 1 ~x6 1 ~x9 >= 2
line 007: p 42 43 +
  ConstraintId 045: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x8 1 x9 1 x10 1 x11 >= 3
...
  ConstraintId 061: 1 ~x5 1 ~x6 1 ~x10 >= 2
line 028: p 53 57 + 59 + 61 +
  ConstraintId 062: 1 x8 1 x11 1 x12 >= 2
line 029: u >= 1 ;
  ConstraintId 063: >= 1
line 030: c -1
=== end trace ===

Verification succeeded.
```

Different Clique Algorithms

Different search orders?

✓ Irrelevant for proof logging.

Using local search to initialise?

✓ Just log the incumbent.

Different bound functions?

- Is cutting planes strong enough to justify every useful bound function ever invented?
- So far, seems like it…

Weighted cliques?

✓ Multiply a colour class by its largest weight.

✓ Also works for vertices "split between colour classes".

What About Subgraph Isomorphism?

Each pattern vertex gets a target vertex:

$$\sum_{t \in \mathrm{V}(T)} x_{p,t} = 1 \qquad\qquad p \in \mathrm{V}(P)$$

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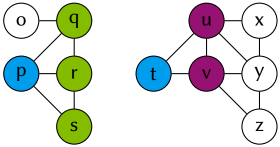
Adjacency constraints, if *p* is mapped to *t*, then *p*'s neighbours must be mapped to *t*'s neighbours:

$$\overline{x}_{p,t} + \sum_{u \in \mathrm{N}(t)} x_{q,u} \geq 1 \qquad\qquad p \in \mathrm{V}(P),\ q \in \mathrm{N}(p),\ t \in \mathrm{V}(T)$$

Degree Reasoning in Cutting Planes

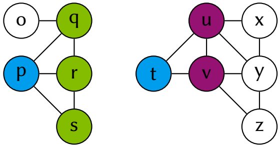


A pattern vertex p of degree $\deg(p)$ can never be mapped to a target vertex t of degree $\deg(p) - 1$ or lower in any subgraph isomorphism.

Observe $\mathrm{N}(p) = \{q, r, s\}$ and $\mathrm{N}(t) = \{u, v\}$.

We wish to derive $\overline{x}_{p,t} \geq 1$.

Degree Reasoning in Cutting Planes



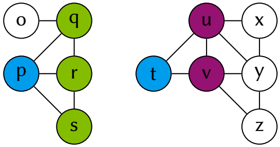
We have the three adjacency constraints,

$$\overline{x}_{p,t} + x_{q,u} + x_{q,v} \geq 1$$
$$\overline{x}_{p,t} + x_{r,u} + x_{r,v} \geq 1$$
$$\overline{x}_{p,t} + x_{s,u} + x_{s,v} \geq 1$$

Their sum is

$$3\overline{x}_{p,t} + x_{q,u} + x_{q,v} + x_{r,u} + x_{r,v} + x_{s,u} + x_{s,v} \geq 3$$

Degree Reasoning in Cutting Planes



Continuing with the sum

$$3\overline{x}_{p,t} + x_{q,u} + x_{q,v} + x_{r,u} + x_{r,v} + x_{s,u} + x_{s,v} \geq 3$$

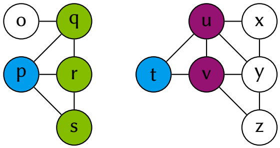
Due to injectivity,

$$-x_{o,u} + -x_{p,u} + -x_{q,u} + -x_{r,u} + -x_{s,u} \geq -1$$
$$-x_{o,v} + -x_{p,v} + -x_{q,v} + -x_{r,v} + -x_{s,v} \geq -1$$

Add all these together, getting

$$3\overline{x}_{p,t} + -x_{o,u} + -x_{o,v} + -x_{p,u} + -x_{p,v} \geq 1$$

Degree Reasoning in Cutting Planes



We're more or less there. We have:

$$3\overline{x}_{p,t} + -x_{o,u} + -x_{o,v} + -x_{p,u} + -x_{p,v} \geq 1$$

Add the literal axioms $x_{o,u} \geq 0$, $x_{o,v} \geq 0$, $x_{p,u} \geq 0$ and $x_{p,v} \geq 0$ to get

$$3\overline{x}_{p,t} \geq 1$$

Divide by 3 to get the desired

$$\overline{x}_{p,t} \geq 1$$

Degree Reasoning in VeriPB

```
p 18$_{p\sim t:q}$ 19$_{p\sim t:r}$ + 20$_{p\sim t:s}$ +    * sum adjacency constraints
  12$_{inj(u)}$ + 13$_{inj(v)}$ +            * sum injectivity constraints
  xo_u + xo_v +                  * cancel stray xo_*
  xp_u + xp_v +                  * cancel stray xp_*
  3 d                            * divide, and we're done
```

Or we can ask VeriPB to do the last bit of simplification automatically:

```
p 18$_{p\sim t:q}$ 19$_{p\sim t:r}$ + 20$_{p\sim t:s}$ +    * sum adjacency constraints
  12$_{inj(u)}$ + 13$_{inj(v)}$ +            * sum injectivity constraints
j -1 1 ~xp_t >= 1 ;              * desired conclusion is implied
```

Other Forms of Reasoning

We can also log all of the other things state of the art subgraph solvers do:

- Injectivity reasoning and filtering.
- Distance filtering.
- Neighbourhood degree sequences.
- Path filtering.
- Supplemental graphs.

Other Forms of Reasoning

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- Path filtering.
- Supplemental graphs.

Proof steps are "efficient" using cutting planes.

- The length of the proof steps are no worse than the time complexity of the reasoning algorithms.
- Most proof steps require only trivial additional computations.

Extension Variables

Suppose we want new, fresh variable a encoding

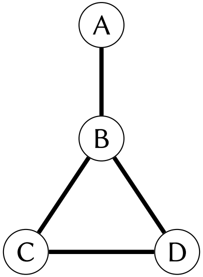
$$a \Leftrightarrow (3x + 2y + z + w \geq 3)$$

Introduce constraints

$$3\overline{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

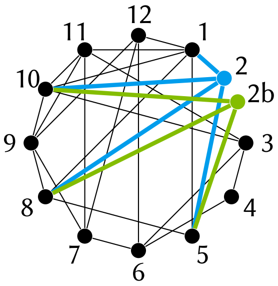
Should be fine, so long as a hasn't been used before.

Symmetries



- If a solution exists, a solution where $C < D$ exists.

Dominance



Can ignore vertex 2b.

- Every neighbour of 2b is also a neighbour of 2.

Progress So Far on World Domination

- SAT with symmetries, cardinality, XOR reasoning, MaxSAT.
  - Uncovered several undetected bugs in state of the art solvers.
  - Can't do MaxSAT hitting set solvers yet, MIP isn't proof logged.
- Certified translations from pseudo-Boolean to CNF.
- Clique, subgraph isomorphism, maximum common (connected) induced subgraph.
- Constraint programming.
  - Large integer variables.
  - Absolute value, all different, circuit, comparison, element, linear equality and inequality, minimum and maximum, regular, smart table constraints.
- In progress: MIP preprocessing for pseudo-Boolean problems, dynamic programming, the remaining 400 constraints for CP, …

What Reasoning Can We Justify?

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- With extension variables, as strong as Extended Frege.
- So according to theorists, we can simulate pretty much everything.
    - Up to a polynomial factor…
- Except dominance is apparently even stronger?

What Reasoning Can We Justify Efficiently?

- Quadratic overheads are unpleasant.
- Cutting planes is very good at justifying combinatorial arguments.
- It's not really clear why.

Verifying the Verifier

- How do we know the encoding is correct?
- How do we know the verifier is correct?
- How do we know the proof system is sound?

Proof Trimming

- Proofs can be really really really big.
- Often many steps end up being redundant for the final proof.
- Could we make a tool that turns a really really really big proof into a really big proof?

Counting and Sampling without Enumerating

- The proof system deals with unsatisfiability.
- Satisfiability is easy, just give a solution.
- Optimisation is a solution and a proof there's nothing better.
- Enumeration is a solution list, and a proof there's nothing else.
- How do we provide a count without enumerating?

Going the Other Way

- Can we use proofs to understand solver behaviour?
  - Why solvers work so well when they shouldn't.
  - Why solvers perform so badly when they shouldn't.
- Explainability?

Where We're At

- Can verify *solutions* from state of the art combinatorial solving algorithms, in a unified proof system.
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- Can verify *solutions* from state of the art combinatorial solving algorithms, in a unified proof system.
- Found many undetected bugs in widely used solvers.
  - Including in algorithms that have been "proved" correct.
- Not being either proof logged or formally verified should be considered socially unacceptable.
- Perhaps studying proof logs can help explain why solvers work so well?

Getting Involved

- Glasgow has funding for PhD students starting this October.
- I will be hiring for a three year postdoc position as soon as the paperwork is finished.
- Install VeriPB: `https://gitlab.com/MIAOresearch/software/VeriPB`
- Documentation: `https://satcompetition.github.io/2023/downloads/proposals/veripb.pdf`
- Tutorial: `https://www.youtube.com/watch?v=s_5BIi4I22w`

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