Finding Maximum k-Cliques Faster Using Lazy Global Domination

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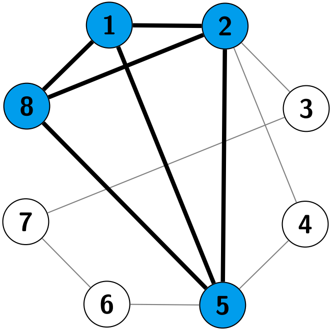
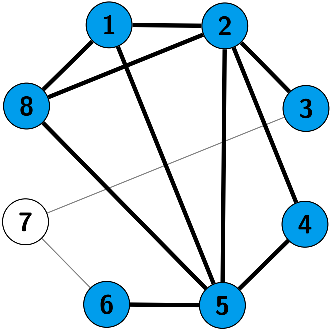
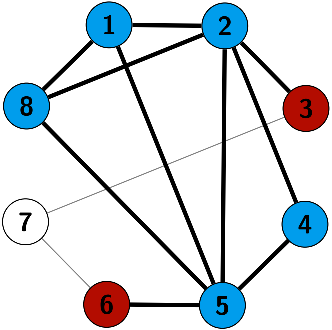
Maximum Clique



Maximum k-Clique



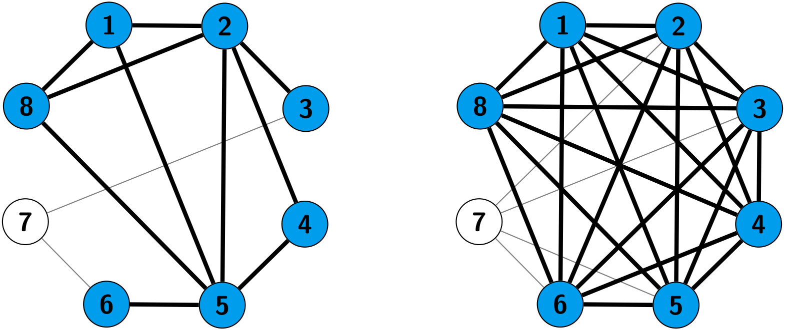
Maximum k-Club



Existing Work

- Many computational papers on the *k*-club problem.
  - Not a hereditary property, and modelling it is hard, so lots of scope for being clever.
- "Unlike the maximum clique problem, the maximum *k*-clique problem has not been the subject of extensive research and we are not aware of any computational results for this problem to date." (Shahinpour and Butenko, 2013)

Reducing k-Clique to Clique



But Is This Practical?

- We probably want to work with large sparse graphs as inputs.
- There are many good maximum clique algorithms for large sparse graphs.
- But G^k is large and dense!
- There are also good maximum clique algorithms for small dense graphs, but small dense graphs can be *very* hard.
  - $\omega(\mathrm{G}(500, 0.9))$ can't be solved in a CPU-decade.
- It's well known in the CP community that reduction to clique is a bad way of solving things.
  - (This is not necessarily true, but it's well known...)

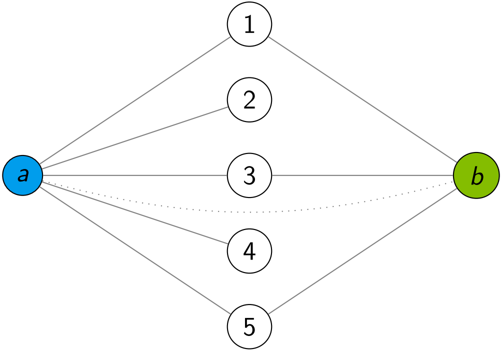
The Good News

- Start with a state-of-the-art maximum clique algorithm for small dense graphs.
- Replace cubic preprocessing and inference with cheaper quadratic algorithms.
- Spend some time making the G^k reduction fast.
- Buy a hefty amount of RAM.

The Results ($k \in \{2, 3, 4\}$)

- Erdős collaboration graphs:
  - $|V| \leq 6{,}927$, $|E| \leq 11{,}850$, $D(G^k) \leq 0.57$ (or $= 1$).
  - 16 take $< 1s$, 2 take $< 12s$, 3 $> 1h$.
- DIMACS 2 Clique Graphs with diameter > 2:
  - $|V| \leq 500$, $|E| \leq 46{,}627$, $D(G^k) \leq 0.87$ (or $= 1$)
  - 22 take $< 1s$, 2 take $> 1h$.
- DIMACS 10 Partitioning Graphs (smallest 20):
  - $|V| \leq 36{,}519$, $|E| \leq 1{,}007{,}284$, $D(G^k) \leq 0.67$.
  - 2 take $> 1h$, most take well under a minute.
- DIMACS 10 Clustering Graphs (smallest 20):
  - $|V| \leq 40{,}421$, $|E| \leq 175{,}691$, $D(G^k) \leq 0.99$.
  - 7 take $> 1h$, most take well under a minute.

But We Can Do Better



Better Results!

- Erdős collaboration graphs:
  - The three open instances now take 3.8s, 69.4s, 207.3s.
- DIMACS 2 Clique Graphs:
  - The two open instances now take 0.1s.
- DIMACS 10 Partitioning Graphs:
  - One open instance now takes 16.9s, the other still takes $> 1h$.
- DIMACS 10 Clustering Graphs:
  - Two open instances closed.
- Not a universal success, though:
  - Factor of 10 slowdown on a few fairly easy instances.
  - Without laziness, the results are terrible. . .

What About Parallel Search?

- Active research topic for maximum clique.
  - Work stealing strategies really make a huge difference (often more so than load balance)!
- In this paper:
  - Dynamic work splitting, starting at the top, and working downwards.
  - Use parallelism to steal early, where value ordering heuristics are weakest—a bit like discrepancy search or restarts.

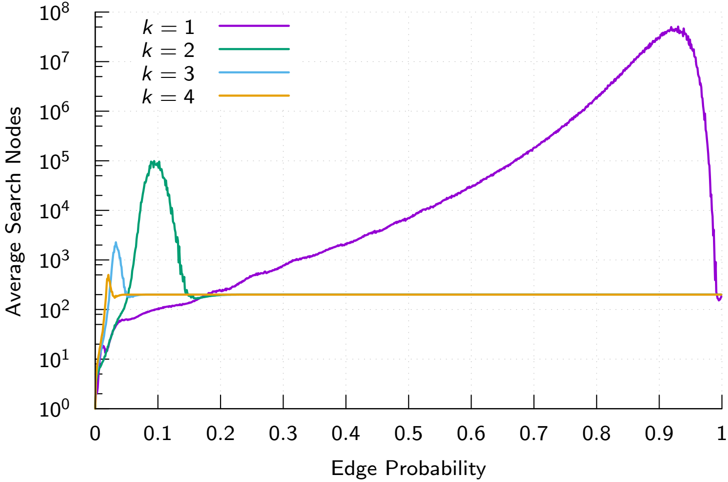
Even Betterer Results!

- Two open instances closed.
- Improved bounds on four remaining instances.
- Also super-linear speedups in (at least) two cases.
- But only if we get it right:
    - Again, work stealing strategies matter.
    - Load balancing is harder: must use a much deeper splitting limit than for typical maximum clique graphs.

Slide of Blatant Propaganda

- This technique is the only known practical way of getting:
  - Reproducible runtimes.
  - Guarantees of no (exponential) slowdown over sequential, or when adding cores.
  - Decent work distribution.
  - Respectable speedups in practice.
- I personally think these properties are all critical for practical multi-core parallel search.

What About Random Graphs?



Conspicuously Missing From This Paper

- Parallel G^k construction.
- Predicting $G(n, p)^k$ on random graphs.
- So just why are G^k graphs so easy to solve, anyway?

Conclusion

- There are many clique relaxations (density-based, degree-based, distance-based, . . .). It's often not clear which you'd want in practice.
- k-Clique tends to be easy to calculate, at least.
- Usually the k-Clique and k-Club numbers are the same, too.
- The domination rule is probably useful in other settings.
- Work-stealing strategies matter when doing parallel search.

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